OPTIMAL FILTERING OF RADAR DATA

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Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ENGINEERING ELECTRONICS

United States Naval Postgraduate School
Monterey, California

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ABSTRACT

An optimal method of filtering radar data is analyzed and experimented.

First the theoretical bases of Kalman's approach to filtering problems is presented. Then the suitability of its conclusions to the filtering of radar data is studied. The form for the filter needed in such a case is derived and, finally, a simulation is made attempting to show the real usefulness of Kalman's filter.

It is concluded that the method considered presents a very promising set of features which may lead to advantageous practical use.

The filtering process is based on the recurrent solution of matrix formulas. To accomplish a reasonably fast computation of the filtered values the solution was programmed in the FORTRAN language and run in a 1604 CDC computer.

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1. Introduction

We will be concerned in this work with the filtering of radar data.

To our knowledge the solution of this specific problem, in a majority of cases at least, was never attempted with the help of a deep theoretical foundation. So far the "α, β Tracker" seems to have conveyed almost all the possibilities of practical processing of radar data.

With the new approach to the general problem of filtering announced by Kalman [1] a new frame of practical processing stood up to the engineer.

The "α, β Tracker" has no inherent means or need of utilizing the known statistics of the targets to be tracked in any mathematical or even rational way.

The Kalman approach starts out by quantifying the known parameters of the signal generating process. Broadly speaking, the class of problems attacked by Kalman consists of filtering and predicting the output of Gauss-Markov random processes when its parameters and statistics are known. The method described by Kalman being particularly suited to machine computation was immediately explored in more than one field of possible applications.

One branch of the work done with the Kalman filter occupied control engineers. Another one gave results in the tracking of artificial satellites.

Rauch [2] and Magill [3] followed Kalman with the analysis of cases in which the process parameters are random variables.

It will be shown that Magill's Technical Report is completely suitable to provide a theoretical basis upon which a radar data filter can be built with fairly good results.

2. Statement of the problem.

The processing of data provided by the various types of radars can be readily implemented with the digital computer.

The processing device must be able to allocate the echoes received to one of the various targets being tracked, to decide on the appearance of a new target, to give numerical significance to the distance, azimuth and other characteristics detected by the radar, to store sequentially the information being obtained and to provide all the data stored to any unit or entity allowed to obtain it.

This data will be considered to be present and available for filtering purposes.

The data being stored needs to be handled in such a way that the various tracks maintained are filtered as accurately as possible. It is also considered that a prediction will have to be made and transmitted to some other system.

The characteristics of the data provided by the computer storage for an eventual solution of the filtering problem are, obviously, of great importance.

Admitting that the allocation of echoes to targets was done previously with total confidence, we do not need to consider but the filtering of one target track.

This way for each target we will have stored data with the following characteristics,

a) Numerical values assigned to distance, azimuth and altitude.

b) Times at which the position was detected.

c) If relevant, data on the position, speed and course of the vehicle where the radar is mounted.

Adding to this data collected in the machine storage all the characteristics of the target defining its dynamic model are also necessary to improve the tracking efficiency. We may specify at this point that it is absolutely necessary to have some "a priori" degree of knowledge about the dynamics of possible targets when the first point of the track is treated in the filtering process. As will be seen, the filter will not start its recurrent computation unless previous information on the characteristics of the possible targets is available.

We need then:

d) A dynamic model capable of describing the motion of the tracked object.

e) Some statistics of the inputs to that model.

Finally, the source of the noise to be filtered out has to be considered. Noise will be introduced by the radar itself, by the processing prior to the filter action, and by any anomalies in the propagation of the electromagnetic waves.

Let us consider then:

f) Statistics of the error introduced by the equipment.

With this data available the filtering process will aim at reproducing the real track of the target, eliminating errors added during the acquisition of data.

We will see that this objective is not completely achieved but that the mean square error of the positions will be reduced to a minimum, with the help of the Kalman filter.

In d) we referred to the need of a dynamic model capable of describing the motion of the object being tracked. More precisely we can say that this model is going to include or use, in one way or another, all the data gathered by us or available in storage. We may even say that

this dynamic model is central to the solution we are seeking. With this in view let us proceed to formalize the model and obtain a more rigorous concept of the problem presented to us.

The signal or message and the noise precesses are given by,

$$S(t+1) = \phi_s(t) S(t) + D_s(t+1) U_s(t+1)$$

$$y(t) = r^T(t) S(t)$$

and,

$$W(t+1) = \phi_w(t) W(t) + D_w(t+1) U_w(t+1)$$

$$n(t) = h^T(t) w(t),$$

Where the various symbols have the following meanings:

$S(t)$ is the present state vector of the message process, i.e., a column matrix formed by the target components of distance, velocity, acceleration, etc., on some coordinate axis (which we assume to be the X-axis from now on). In chapter three we will indicate how the other components can be treated by the same filter.

$\phi_s(t)$ is the transition matrix of the message process, interpreted as the matrix by which we must multipy the present state to obtain the next, as will clearly be seen in the end of this chapter. This includes the knowledge of d).

$U_s(t)$ is a vector that includes all the possible inputs to the message process, as wind gusts (which we will consider as being able to change instantly the position of the target), air speed and variations of acceleration as caused by the pilot's action. As later will be seen, these inputs have to be white gaussian random variables in order that the theory can be applied.

$D_s(t)$ is the distribution matrix of the message process and determines in which way the inputs affect the various state vector components.

$y(t)$ is the message output which may consist of one or more of the state vector components.

$r(t+1)$ is the output vector of the message process which determines what state vector components are going to be used in the model output.

$\phi_w(t)$ is the transition matrix of the noise process.

$U_w(t)$ is the white noise process vector. Includes the knowledge of f).

$n(t)$ is the noise process output.

$h(t)$ is the output vector of the noise process.

The central problem of this investigation can now be clearly defined. The information received by the radar and sequentially stored is the sum,

$$z(t) = y(t) + n(t), \qquad t = 1, 2, 3, \ldots\ldots,$$

where $y(t)$* represents the true distance and $n(t)$ the error added to that. We will seek a procedure of machine computation to obtain $y(t)$ at each instant $t=1,2,3,\ldots$ and, in a more general fashion, to obtain some $\hat{y}(t) = y(t) + e_y(t)$ where the error $e_y(t)$ is minimized in accordance with the mean-square error criterion.

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* Notice here that as we are treating the X-axis components this will have to be interpreted as distance along the X-axis. This apparent confusion could be avoided but we decided to maintain completely Magill's notation to provide some unity between his report and this application.

3. Consideration of the theory

Later on we will study Kalman filter's adequacy to radar data filtering and will concentrate in the areas of application. But, obviously, a clear understanding of the theoretical results to be used has to be provided. This is the present chapter's objective.

The elements required for a solution may be concentrated in the four equations presented in the last chapter which describe the message and noise processes.

A simplification of this can be made and consists of joining together the message and noise processes, reducing these to a single process which has as output the sum $z(t) = y(t) + n(t)$ This is accomplished by augmenting the state vector.

According to this we now define:

$$x(t) \triangleq \begin{pmatrix} s(t) \\ \hline w(t) \end{pmatrix}$$ - state vector of the observable process.

$$\phi(t) \triangleq \left(\begin{array}{c|c} \phi_s(t) & 0 \\ \hline 0 & \phi_w(t) \end{array}\right)$$ - transition matrix of the observable process.

$$\Delta(t) \triangleq \left(\begin{array}{c|c} D_s(t) & 0 \\ \hline 0 & D_w(t) \end{array}\right)$$

$$v(t) \triangleq \begin{pmatrix} U_s(t) \\ \hline U_w(t) \end{pmatrix}$$

$$m^T(t) \triangleq \left(\begin{array}{c|c} r^T(t) & h^T(t) \end{array}\right)$$ - output vector of the observable process.

Defining also,

$$E\{v(t)v^T(t)\} \triangleq Q(t)$$ - covariance matrix of the inputs.

$$U(t) \triangleq Q^{-\frac{1}{2}}(t)v(t) \;*$$

$$D(t) \triangleq \Delta(t)Q^{\frac{1}{2}}(t)$$

The model for the process which generates the observed signal may now be reduced to the set of equations:

$$x(t+1) = \phi(t)\,x(t) + D(t+1)\,U(t+1)$$
$$z(t) = m^T(t)\,x(t) \,,$$

where $U(t+\tau) = U(t)$, $0 \leq \tau < 1$

Even if this group of definitions casts some shadow on the physical meaning of the parameters we will not try to clarify it right now. The next chapter will provide full understanding by means of an example.

We may now represent the whole process in a block-diagram form, Fig. 1, where the parallel lines represent flow of vectors instead of a scalar value as in the line at right for $z(t)$.

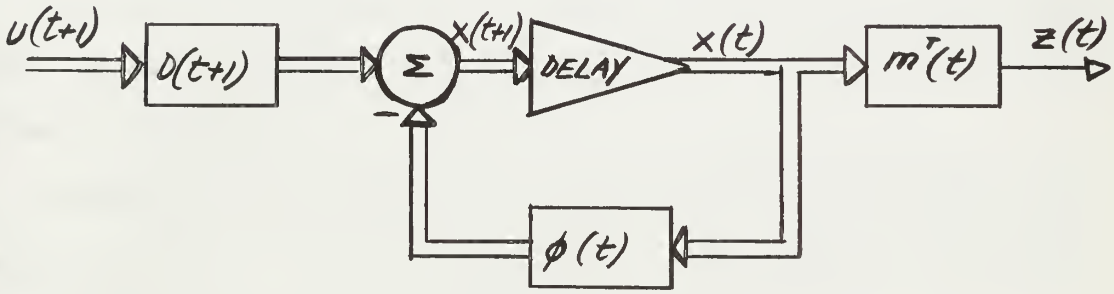

Fig. 1 Model of the observable process

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*As $Q(t)$ is a diagonal matrix, $Q^{-\frac{1}{2}}(t)$ is equivalent to apply the exponent $-\frac{1}{2}$ to all non-zero elements of this matrix.

Let us proceed now to look at the solution of the filtering problem.

The objective is to find a recurrence formula to obtain for each sampling point the best estimate of the state vector, given both the previous one and observed state vectors at that sampling point.

The best estimate of the state vector we need is by definition,

$$\hat{x}(t/t-1) \triangleq E\{x(t)/Z_{t-1}\},$$

Where Z_{t-1} represents all the available data up to and including time $t-1$.

$$\tilde{z}(t/t-1) \triangleq z(t) - \hat{z}(t/t-1),$$

where $z(t)$ is the observed signal at time t and $\hat{z}(t/t-1)$ is the best estimate of the observed signal at time t given the previous best estimate. It is possible to derive [3] a recurrence formula,

$$\hat{x}(t/t) = \hat{x}(t/t-1) + K(t,t)\,\tilde{z}(t/t-1) \quad , t=1,2,3,\ldots , \tag{1}$$

where $K(t,t)$ represents a variable gain vector.

We now observe that the filter produces the best estimate at time t given the data collected up to and including that time by means of the following operations,

a) Evaluates a best estimate at time t given data collected up to and including $t-1$. This is simply done by multiplying the transition matrix $\phi(t-1)$ by the best estimate $\hat{x}(t-1/t-1)$.

b) Corrects $\hat{x}(t/t-1)$ by the addition to it of a term containing the information received at t . This term is the product of a weighting coefficient - the gain $K(t,t)$ - and the error in estimating the observed signal given data collected up to and including $t-1$.

From what has been said we understand that the filter is essentially formed with a model of the message process and the added feature of being able to correct the updating of its output with the information received.

The recurrence formula (1) is the solution to the filtering problem. Of course the means of updating the gain matrix is still not uncovered.

The derivation of a recurrence formula for this matrix starts out considering that the best estimate can be obtained by minimizing the trace of the following matrix,

$$P(t/t) \stackrel{\Delta}{=} E\{\tilde{x}(t/t)\tilde{x}^T(t/t)\}$$

where $\tilde{x}(t/t) \stackrel{\Delta}{=} x(t) - \hat{x}(t/t)$,

$X(t)$ being the true state vector of the message and noise generated.

The matrix $P(t/t)$ is the covariance matrix of the estimation error and, obviously, gives an indication of the error value at any t.

Further defining,

$$\sigma^2(t/t-1) \stackrel{\Delta}{=} m^T(t)\,P(t/t-1)\,m(t) = \mathrm{var}\{\tilde{z}(t/t-1)\} \qquad (2)$$

it is possible to obtain

$$k(t,t) = \frac{P(t/t-1)\,m(t)}{\sigma^2(t/t-1)} \qquad (3)$$

A recurrence formula for $P(t/t-1)$ must still be obtained. Again Magill derives it based on the assumptions previously made and reaches the result,

$$P(t+1/t) = \phi(t)\left[I - k(t,t)\,m^T(t)\right]P(t/t-1)\,\phi^T(t) + D(t)D^T(t) \qquad (4)$$

The recurrent procedure is now completely obtainable, provided the initial covariance matrix $P(t_0/$ previous knowledge$)$ is given. This is done by sequentially evaluating $k(t,t), \hat{x}(t/t), P(t+1/t)$, etc..

With the equations we have introduced, a filter block-diagram is possible.

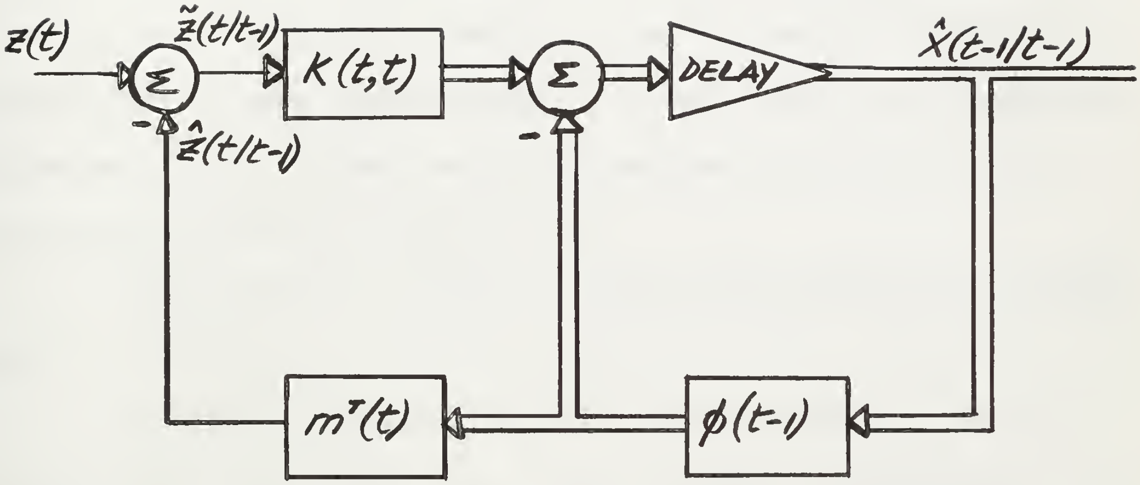

Fig. 2 Model of the optimal filter

4. Applications

In this chapter we intend to analyze more closely the assumptions made before getting the results of the last chapter and test them in the problem of filtering radar data. We will then decide on the applicability to our problem of the theoretical results presented before.

As it was said, all Magill's derivations are based on the fact that the processes involved are either Gauss-Markov or can be obtained from these by application of a linear matrix multiplier. Can the state vector of any flying target, whatever dynamic properties it may have, be considered a Gauss-Markov process? In fact, given any instantaneous state vector, it is always possible to consider some other previous time when the definition applies, i.e.,

$$p[x(t+1)/x(t), x(t-1), \ldots] = p[x(t+1)/x(t), x(t-1), \ldots, x(t-k)],$$

where k is finite.

In a physical sense we may say that if we want, at any instant t, to obtain the state vector at $t+1$, we may discard the knowledge we have of the target's state vector before the instant $t-k$. For example, if we consider an aircraft at t, clearly it is seen that the knowledge of all its positions since the flight started is normally irrelevant to determine the next one but we are almost certainly interested in the last two or three. Of course the number k will vary with the type of target considered but in any case is a finite number.

In Chapter two we said in d) that we would need a dynamic model for the tracked target. This model and the information of e) (some statistics of the inputs) and f) (statistics of the radar error) were all considered in the formulas then presented.

Let's analyze first the message process,

$$s(t+1) = \phi_s(t)\, s(t) + D_s(t+1)\, U_s(t+1)$$
$$y(t) = r^T(t)\, s(t)$$

If we consider in the state vector only distance, speed and accelleration and this way eliminate the possibility of a third derivative in the motion we have,

$$\begin{bmatrix} s_1(t+1) \\ s_2(t+1) \\ s_3(t+1) \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) & \phi_{13}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \phi_{23}(t) \\ \phi_{31}(t) & \phi_{32}(t) & \phi_{33}(t) \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} + \begin{bmatrix} D_{s11}(t+1) & D_{s12}(t+1) & D_{s13}(t+1) \\ D_{s21}(t+1) & D_{s22}(t+1) & D_{s23}(t+1) \\ D_{s31}(t+1) & D_{s32}(t+1) & D_{s33}(t+1) \end{bmatrix} \begin{bmatrix} U_{s1}(t+1) \\ U_{s2}(t+1) \\ U_{s3}(t+1) \end{bmatrix}$$

Simplifying the notation and multiplying matrices we obtain,

$$s_1 = \phi_{11} s_1 + \phi_{12} s_2 + \phi_{13} s_3 + D_{s11} U_{s1} + D_{s12} U_{s2} + D_{s13} U_{s3}$$

The first three terms of the second member of the equation represent the formula to obtain the distance at some instant $t+1$, the distance, velocity and acceleration at t being known. The components ϕ_{11}, ϕ_{12} and ϕ_{13} could be easily gotten through the Taylor series expansion and as we know they are

$$\phi_{11} = 1\ ,\ \phi_{12} = T\ ,\ \phi_{13} = \tfrac{1}{2} T^2\ ,$$

where T is obviously the sampling period.

The other components of the ϕ matrix are easily obtained in the same fashion.

We can now give full significance to $D_s(t+1)U_s(t+1)$ if we recall the definition of the $U_s(t+1)$ matrix given before. It is simply the way of correcting the next value of the target's state vector by adding the possible random inputs. For example, we may obtain the target's next velocity value if we know the present one, the acceleration and the sampling period but notice that we still have to consider the introduction, during the sampling period, of some wind velocity and then we have to complete the value of $s_2(t+1) = s_2(t) + T s_3(t)$ by the addition

of $U_{s2}(t+1)$.

The random inputs $U_{s1}(t+1)$ and $U_{s3}(t+1)$ account for instantaneous changes of position and for variations of acceleration introduced by differences in the engine's thrust.

All put together in the matrix equation we have,

$$\begin{bmatrix} s_1(t+1) \\ s_2(t+1) \\ s_3(t+1) \end{bmatrix} \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_{s1}(t+1) \\ U_{s2}(t+1) \\ U_{s3}(t+1) \end{bmatrix}$$

One feature of the random inputs when considered together is that one of them may substitute the entire vector. Consider, for example, $U_{s3}(t+1)$. Through this acceleration input we can account for any variation in position or velocity which instantaneously may happen.

Looking now at the second equation of the message process and considering $y(t)$ as a vector containing only distance,

$$[r(t)] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In the noise process we have a different case. We consider the noise at any sampling time to be independent from any other noise value.

This makes all the elements in $\phi_w(t)$ null.

Let us make it clear also that the state vector $W(t+1)$ has only one component which is, obviously, the error introduced in the measurement of distance, the only one made. The input $U_w(t+1)$ is then the actual error $W(t+1)$.

We have then,

$$[w(t+1)] = [0][w(t)] + [1][U_w(t+1)], \tag{5}$$

and,

$$[n(t)] = [1][w(t)] \tag{6}$$

A point we still did not look at is the gaussian quality of the random inputs. Can we admit that the acceleration imparted to an aircraft when considered in discrete amounts constitutes a random variable with gaussian distribution and zero mean? If we think of all the aircraft and all its flights we, naturally, would be ready to accept that. And actually, this is what happens once the filtering must be appropriate for any aircraft that can appear in the radar scope. This has to be kept in mind when values are going to be given to the inputs. We will return to this problem in the next chapter.

Having formed the message and noise processes we may now proceed to construct the model for the observable process, as explained in Chapter 2.

So,

$$x(t)= \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ w(t) \end{bmatrix}$$

$$\phi(t)= \begin{bmatrix} 1 & T & \frac{1}{2}T^2 & 0 \\ 0 & 1 & T & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta(t)= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v(t)= \begin{bmatrix} U_{s1}(t) \\ U_{s2}(t) \\ U_{s3}(t) \\ U_w(t) \end{bmatrix}$$

$$m^T(t)= \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$$

and with simplified notation,

$$Q(t) \stackrel{\Delta}{=} E \begin{bmatrix} U_{s1}^2 & U_{s1}U_{s2} & U_{s1}U_{s3} & U_{s1}U_{sw} \\ U_{s1}U_{s2} & U_{s2}^2 & U_{s2}U_{s3} & U_{s2}U_{sw} \\ U_{s1}U_{s3} & U_{s2}U_{s3} & U_{s3}^2 & U_{s3}U_{sw} \\ U_{s1}U_{sw} & U_{s2}U_{sw} & U_{s3}U_{sw} & U_{sw}^2 \end{bmatrix}$$

As the random variables $U_{s1}, U_{s2}, U_{s3}, U_{sw}$ have zero mean, we see that,

$$E\{U_{s1}^2\} = Var\, U_{s1} \ ,$$

and,

$$E\{U_{s1}U_{s2}\} = Cov(U_{s1}, U_{s2}) = 0$$

since U_{s1}, U_{s2}, etc. are uncorrelated.

$$D(t) \stackrel{\Delta}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (Var\, U_{s1})^{1/2} & 0 & 0 & 0 \\ 0 & (Var\, U_{s2})^{1/2} & 0 & 0 \\ 0 & 0 & (Var\, U_{s3})^{1/2} & 0 \\ 0 & 0 & 0 & (Var\, U_{sw})^{1/2} \end{bmatrix} =$$

$$\begin{bmatrix} \sigma U_{s1} & 0 & 0 & 0 \\ 0 & \sigma U_{s2} & 0 & 0 \\ 0 & 0 & \sigma U_{s3} & 0 \\ 0 & 0 & 0 & \sigma U_{sw} \end{bmatrix} ;$$

where σ represents the standard deviation.

With the matrix just obtained for $D(t)$ we are able to use the recurrence formulas presented in the preceeding chapter.

But we still need the initial covariance matrix without which we cannot start the computation.

It can be observed very easily that at t_0, i.e., when the first measurement of the observable process is made, the correlations between distance and speed errors, distance and acceleration errors, etc., are all null. We have then,

$$P(t_0/\text{previous knowledge}) = \begin{bmatrix} Var\, \tilde{X}_1 & 0 & 0 & 0 \\ 0 & Var\, \tilde{X}_2 & 0 & 0 \\ 0 & 0 & Var\, \tilde{X}_3 & 0 \\ 0 & 0 & 0 & Var\, \tilde{X}_4 \end{bmatrix}$$

Of course the diagonal values must be based on the information about the target which we have even before we begin tracking it. To avoid a purely conjectural evaluation of the variance we may simply discard the three first received positions from the radar, as far as filtering is concerned, and make use of them to estimate the velocity and acceleration of the target. At t_0+2 we will have then the position, the velocity and the acceleration of the target with errors the variance of which we can determine.

Let us see what are those variances. Supposing that we know the variance of the error in the distance measurement:

By definition $\tilde{x}_1(t_0+2) \stackrel{D}{=} x_1(t_0+2) - \hat{x}_1(t_0+2)$

Then $Var\ \tilde{x}_1(t_0+2) = Var\left[x_1(t_0+2) - \hat{x}_1(t_0+2)\right]$

If we take $\hat{x}_1(t_0+2)$ as the observed value, then we have,

$$Var\ \tilde{x}_1(t_0+2) = Var\ U_{sw}$$

because the error in the distance measurement is the error introduced by the radar.

We want now $Var\ \tilde{x}_2(t_0+2)$. Using the last two points we have, using the same assumptions as above,

$$\tilde{x}_2(t_0+2) = \frac{x_1(t_0+2) - x_1(t_0+1)}{T} - \frac{\hat{x}_1(t_0+2) - \hat{x}_1(t_0+1)}{T} =$$

$$= \frac{x_1(t_0+2) - \hat{x}_1(t_0+2) - \left[x_1(t_0+1) - \hat{x}_1(t_0+1)\right]}{T} =$$

$$= \frac{\tilde{x}_1(t_0+2) - \tilde{x}_1(t_0+1)}{T}$$

Finally $$Var\ \tilde{x}_2(t_0+2) = \frac{1}{T^2}\ Var\left[\tilde{x}_1(t_0+2) + \tilde{x}_1(t_0+1)\right] =$$

$$= \frac{2}{T^2}\ Var\ U_{sw}$$

Let us obtain now

$$Var\ \tilde{x}_3(t_0+2)$$

The three points we have permit us to write,

$$\tilde{X}_3(t_0+2) = X_3(t_0+2) - \hat{X}_3(t_0+2) =$$

$$= \frac{X_2(t_0+2) - X_2(t_0+1)}{T} - \frac{\hat{X}_2(t_0+2) - \hat{X}_2(t_0+1)}{T} =$$

$$= \frac{\frac{X_1(t_0+2) - X_1(t_0+1)}{T} - \frac{X_1(t_0+1) - X_1(t_0)}{T}}{T} - \frac{\frac{\hat{X}_1(t_0+2) - \hat{X}_1(t_0+1)}{T} - \frac{\hat{X}_1(t_0+1) - \hat{X}_1(t_0)}{T}}{T} =$$

$$= \frac{X_1(t_0+2) - 2X_1(t_0+1) + X_1(t_0)}{T^2} - \frac{\hat{X}_1(t_0+2) - 2\hat{X}_1(t_0+1) + \hat{X}_1(t_0)}{T^2}$$

Now

$$\text{Var}\,\tilde{X}_3(t_0+2) = \frac{1}{T^4}\left[\text{Var}\,\hat{X}_1(t_0+2) + 4\,\text{var}\,\hat{X}_1(t_0+1) + \text{Var}\,\hat{X}_1(t_0)\right] =$$

$$= \frac{6}{T^4}\,\text{Var}\,U_{sw}$$

5. Computation of the filtering process

In the beginning of this work we showed our interest in the eventual machine computation of the recurrent process. It is now time to look at the procedure introduced in the last chapters and try to evaluate how effective the machine computation can be made.

In accordance with what we mentioned in chapter three the formulas below constitute the recurrent process,

$$P(t/t-1) = \phi(t-1)\left[I - k(t-1,t-1)m^T(t-1)\right]P(t-1/t-2)\phi^T(t-1) + b(t-1)b^T(t-1)$$

$$\sigma^2(t/t-1) = m^T(t)\,P(t/t-1)\,m(t)$$

$$k(t/t) = \frac{P(t/t-1)\,m(t)}{\sigma^2(t/t-1)}$$

$$\tilde{z}(t/t-1) = m^T(t)\phi(t-1)\hat{x}(t-1/t-1)$$

$$\hat{x}(t/t) = \phi(t-1)\hat{x}(t-1/t-1) + \tilde{z}(t/t-1)k(t,t)$$

The program for this computation can be done very easily. We have in the next page a very simple flow-chart that illustrates this array of computations.

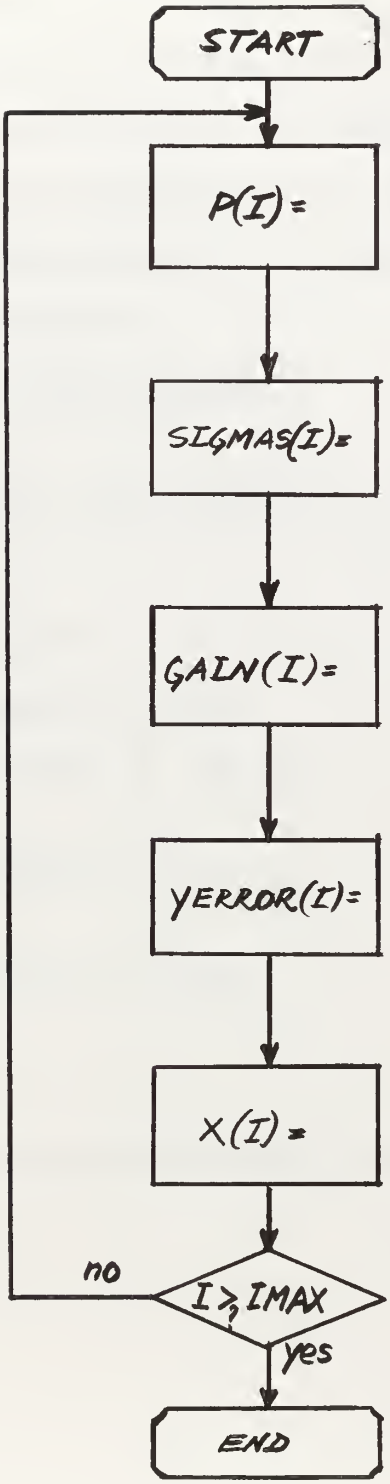

Fig. 3 Flow-chart for recurrence formulas

The problem of how to start the recurrent process has still to be considered. We said before that we could obtain the value of the initial covariance matrix by simply taking the three first points of the raw track, evaluating with them the position, velocity and acceleration at the third point and, at the same time, calculate the initial covariance matrix of the state vector thus obtained.

The formulas then mentioned are repeated here for convenience,

$$\hat{X}_1(t_0+2) = Y(t_0+2)$$

$$\hat{X}_2(t_0+2) = \frac{Y(t_0+2)-Y(t_0+1)}{T}$$

$$X_3(t_0+2) = \frac{\frac{Y(t_0+2)-Y(t_0+1)}{T} - \frac{Y(t_0+1)-Y(t_0)}{T}}{T}$$

$$X_4(t_0+2) = 0$$

$$\mathrm{Var}\,\tilde{X}_1(t_0+2) = \mathrm{Var}\,U_{sw}$$

$$\mathrm{Var}\,\tilde{X}_2(t_0+2) = \frac{2}{T^2}\,\mathrm{Var}\,U_{sw}$$

$$\mathrm{Var}\,\tilde{X}_3(t_0+2) = \frac{6}{T^4}\,\mathrm{Var}\,U_{sw}$$

$$\mathrm{Var}\,\tilde{X}_4(t_0+2) = \mathrm{Var}\,U_{sw}$$

Considering the computations above we obtain the flow diagram on the next page.

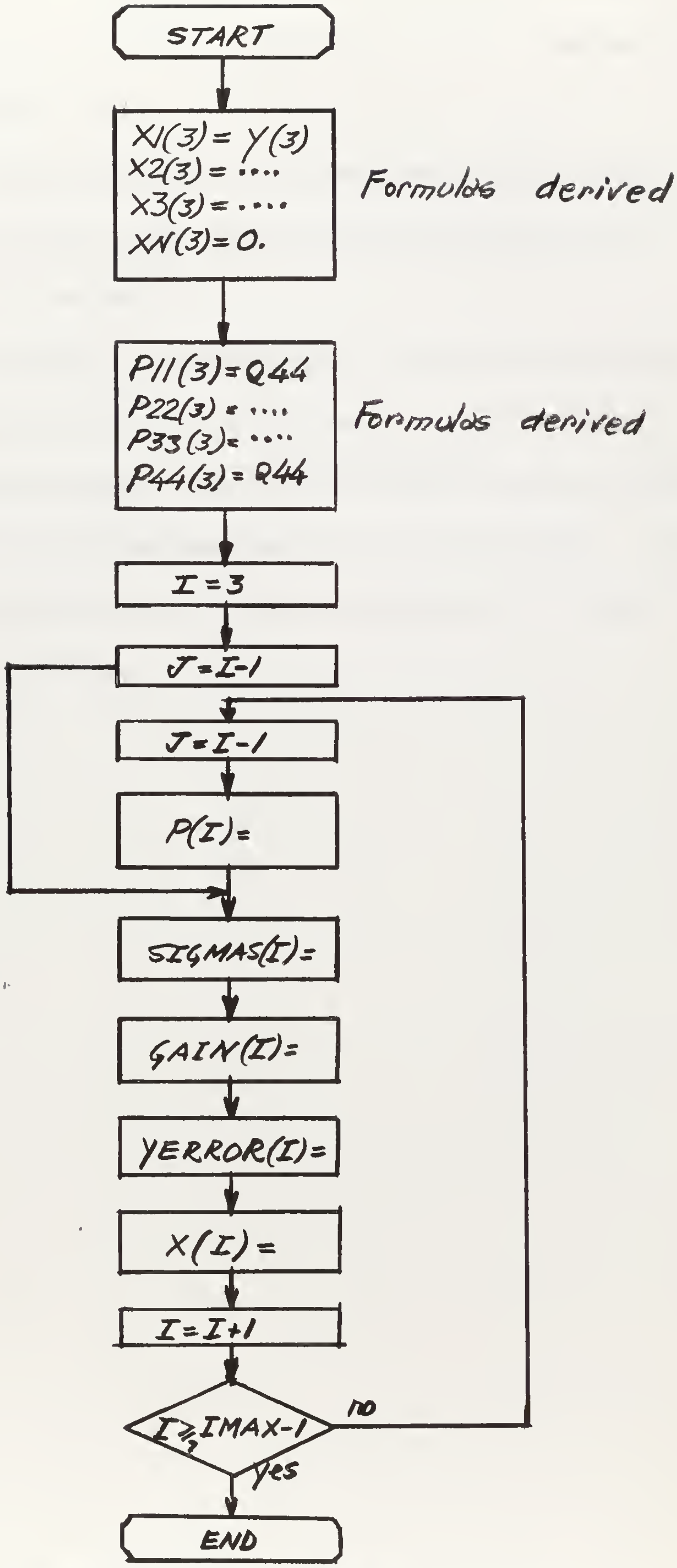

Fig. 4 Flow-chart for the optimum filter

This way we use the initial covariance matrix at t_0+2 $(I=3)$ to obtain the next one at t_0+3 and use the observed positions at t_0+2 to obtain a best estimate at t_0+3 . Our filter will then begin filtering at and including t_0+3 .

To avoid too long statements which would add only to confusion in the program we divided the computation of the covariance matrix in parts as it can easily be seen.

Another point in need of consideration is the way the program is implemented without the help of sub-routines, even if some matrix operations are repeated several times. We thought that some computing time would be saved by eliminating the call of sub-routines. This way there is no great compilation time spent and probably the compilation done comes out more efficient.

6. Example of a simplified simulation

In this chapter we are finally ready to implement the techniques which have been studied until now. Ideally that should be done in a real case. A seemingly feasible way would be to obtain radar data from maneuvering aircrafts, project the points in the X, Y and Z axis and try to filter the three tracks thus obtained. This way of implementing our methods is, however, not too flexible.

A better way is to create a track, add some noise and try to filter this out afterwards. This approach will provide us with the possibility of varying the track at our will and thus permit the evaluation of the filter's suitability in various conditions. How much the results can be relied upon depends, obviously, on the simulation's fidelity. This we will try to assess in the next chapter.

As we have said before, to study our filtering problem it is irrelevant to consider more than the projection of the target's track on one of the coordinate axis. The procedure for the other axis is exactly the same.

Limiting ourselves to the projection on the X-axis we can have three different situations - constant velocity, constant acceleration and sinusoidally varying distance. These correspond in three dimensions to constant velocity, constant acceleration and circular track. Any other track of irregular form can be reduced to a group of circles with accuracy as high as we want.

It seems then only reasonable that we study tracks where these three possibilities happen in different orders and with different values. From here we may eventually conclude that the filter provides acceptable results in any conceivable simulated situation.

Furthermore it seems logically acceptable to conclude the following:

a) The filter will yield good results with any constant velocity given if that is the case for the lowest and highest expected velocities simulated.

b) The filter will work in good conditions for any constant acceleration given that it works for the highest possible and fulfills the requisite a).

c) The filter will definitely accept any simulation and yield good results if, furthermore, it gives acceptable results with the tightest circular maneuver possible - in the sense that the acceleration on the X-axis reaches the highest value possible with the speeds available.

At this point we will have to decide what types of targets are we going to be handling. Let us confine ourselves to airplanes.

To simulate a track we will suppose an aircraft going through all the situations presented above.

Furthermore it seems desirable to have all the combinations of those pieces of track that seem to introduce different degrees of difficulty.

The tightest maneuver we want to explore is the one which gives the greatest maximum acceleration of the motion's projection on the X-axis. It is easy to see by Fig. 2-29 in one of the references [4] that the highest speed and greatest bank angle will give the largest value for acceleration. Considering that the maximum possible maneuvering speed for the aircraft is 1000 knots and the banking angle 70 degrees, we have that the projection on the X-axis of a circle described in the X-Y plane is,

$$X = R \cos(Rot \times t)$$

where R is the radius of turn in miles, Rot the rate of turn in degrees

per second.

Extrapolating in the tables for 1000 mi we have,

$$R = 32{,}000 \ ft = \frac{32{,}000}{6{,}000} ft = 6.4 \ mi$$

$$Rot = 3.0 \times \frac{3.14}{180} = .05237 \ rad/sec$$

Then the maximum acceleration is,

$$X''/_{max} = R \cdot (Rot)^2 = 6.4 \times (.05237)^2 \ mi/sec^2 =$$

$$= 6.4 \times .00275 \ mi/sec^2 = .0176 \ mi/sec^2$$

Let us assume that in a straight line the maximum acceleration possible is 1.805 g's. This gives,

$$1.805 \times 32.2 \ ft/sec^2 = \frac{59.0}{6 \times 10^3} \ mi/sec^2 = 9.83 \times 10^{-3} mi/sec^2$$

To have the largest range of speeds we will consider that our aircraft can fly from 10 to 1000 mi/hr, which is unreal but will give greater scope to our results.

Let us see now what types of tracks will be worthwhile to experiment with. Below are the different combinations which seem to present different degrees of difficulty to our filter.

a) Lowest speed, acceleration to highest speed, highest speed, circle at highest speed.

b) Lowest speed, acceleration to highest speed, circle at highest speed, highest speed.

c) Inverse sequence of a).

d) Inverse sequence of b).

Another factor that may be important is the time during which the aircraft will stay at the lowest and highest speeds.

Two problems are still not solved. One is the addition of noise to the track we are going to create. The other is the sampling interval which corresponds to the radar rotation period.

The noise will be considered to have a Gaussian distribution with a variance of four miles constant for any distance. Of course this is a rather strong simplification and in reality the variance will depend very much on the distance. We do not think that it is too important to deal with this problem right now. In the next chapter this will be discussed again.

The sampling period will vary between some limits in a real situation. But, again, that fact does not seem to be important in this simulation.

The programming of the radar data simulation is very clear and can be analyzed in the Appendix.

First, a time array is generated, $TIME(k)$. Then, for each piece of track we proceed as follows:

For each sampling point the noise is generated with the RNDEV sub-routine which provides us with random numbers obeying to a gaussian distribution with variance of unity. DEV is the output number, which we multiply by $\sqrt{COE}$ to get a variance of valve COE. Then we evaluate the signal output vector using formulas of elementary physics. Finally, the distance is added to the noise giving the input to our filter.

A last question is what values to give to the Q matrix. First of all we are going to assume that all the random inputs are null except the acceleration. Then we will try, by the use of several values for $Q33$, to obtain one which fits best the whole filtering for all the cases we are going to experiment with.

The values that finally were reached are printed out in the Appendix.

To help in the evaluation of the filtering quality a smoothness index was defined as,

$$smooth = \frac{Var(XNT) - Var(XIE)}{Var(XNT)} \times 100\%$$

7. Conclusions and acknowledgements

The results obtained indicate that the Kalman filter can be used in the filtering of radar data with some success.

The cases taken as examples did indicate that in a wide range of real situations the filter will provide useful filtering, as far as the simplifications made are not the reason why the filter worked.

The constant sampling period which was used could have been varied along the track without any difficulty. To prove this we only note that for each computation made after the arrival of a new radar position the value used for T is not related in any way to the last value of T.

The radar error may have a bias depending on the distance. This bias has to be experimentally determined and its value subtracted from the range measured. If not it will eventually introduce a bias in the filtered values.

The Gaussian quality of the radar error is a point of doubt. Only in the presence of a definite radar system can that be asserted. However in a general way we observe that all components of the radar noise seem to have a Gaussian quality.

The value we should give to the Q matrix is critical. Ideally, for each type of target (helicopters, airplanes, missiles, ships, etc.) we should have a Q matrix. The values for $Q33$ can be chosen experimentally. The filter will work with less error when the variance of random inputs is small, as it would be the case of a missile when compared with an aircraft, especially if we consider a ballistic missile.

The components of the motion on the three coordinate axis can obviously be evaluated in succession and with a normal speed computer the computation will be completed before the next set of data arrives.

Before terminating we would like to express our appreciation for the help given during our work by Dr. Harold A. Titus and LCDR Frederick D. Jardine. The first was always ready to clarify, challenge and ultimately give assurance to our ideas and the second worked very closely with us during the study of the theoretical foundations of the Kalman filter.

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2. Rauch, H.E. Linear estimation of sampled stochastic processes with random parameters. Report SEL-62-058 (Tr No. 2108-1), Stanford Electronics Laboratories, Stanford, Cal., April 1962.

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APPENDIX

OPTIMAL FILTER PROGRAM

```
      PROGRAM FILTER 3
      DIMENSION Y(300),G1(300),G2(300),G3(300),G4(330),YERROR(300),
     2X1(300),X2(300),X3(300),XN(300),TIME(300),ITITLE(12),P11(300),
     3P12(300),P13(300),P14(300),P21(300),P22(300),P23(300),P24(300),
     4P31(300),P32(300),P33(300),P34(300),P41(300),P42(300),P43(300),
     5P44(300),XNT(300),X1T(300),X2T(300),X1E(300),X2E(300),X1ES(300),
     6AVX1E(300),VAX1ES(300),VAX1E (300),AVXNT(300),VAXNT(300),
     7XNTS(300),VAXNTS(300),SMOOTH(300),YERRS(300),AVYERR(300),
     8VAYERRS(300),VAYERR(300),ABSXNT(300),ABSX1E(300)
C          THE NEXT CARD READS COE WHICH IS THE SQUARE ROOT OF THE
C     MEASUREMENT NOISE VARIANCE.
      READ 709,COE
  709 FORMAT(F10.0)
      READ 2,         P12(3),P13(3),P14(3),
     2         P21(3),        P23(3),P24(3),
     3         P31(3),P32(3),        P34(3),
     4         P41(3),P42(3),P43(3)
    2 FORMAT(12F5.0)
      READ 3,Q11,Q22,Q33,Q44
    3 FORMAT(4F20.0)
C          THE NEXT CARD READS THE SAMPLING PERIOD.
      READ 5,T
    5 FORMAT (F10.0)
C          THE NEXT CARD READS THE INDEXES WHICH LIMIT THE VARIOUS
C     PIECES OF TRACK.
      READ 300,IMAX1,IMIN2,IMAX2,IMIN3,IMAX3,IMIN4,IMAX4
  300 FORMAT (7I10)
C          THE NEXT CARD READS IN SUCCESSION THE INITIAL DISTANCE
C     AND SPEED AND THE ACCELLERATIONS IN THE FOUR PIECES OF TRACK.
      READ 303,X1T0,X2T0,X3T01,X3T02,X3T03,X3T04
  303 FORMAT(6F10.0)
C          THE NEXT CARD READS THE RADIUS OF ROTATION AND THE RATE
C     OF TURN FOR THE CIRCULAR PIECE OF TRACK.
      READ 705,R,ROT
  705 FORMAT (2F10.0)
      PRINT 999
```

Cont.

```
999 FORMAT (1H1)
    PRINT 708
708 FORMAT(/////10H       COE //)
    PRINT 707,COE
707 FORMAT(F10.2)
    PRINT 800
800 FORMAT (/////7H       T//)
    PRINT 36,T
 36 FORMAT (F10.2)
    PRINT 301
301 FORMAT (/////71H       IMAX1      IMIN2      IMAX2      IMIN3      IMAX
   13      IMIN4      IMAX4//)
    PRINT 302,IMAX1,IMIN2,IMAX2,IMIN3,IMAX3,IMIN4,IMAX4
302 FORMAT (7I10)
    PRINT 304
304 FORMAT (/////72H       X1T0        X2T0        X3T01         X3TC2
   2   X3T03       X3T04   //)
    PRINT 305,X1T0,X2T0,X3TC1,X3T02,X3T03,X3T04
305 FORMAT (6F12.8)
    PRINT 306
306 FORMAT (/////28H        R              ROT    //)
    PRINT 307,R,ROT
307 FORMAT (2F14.8)
    PRINT 706
706 FORMAT (/////56H        Q11            Q22            Q33
   2Q44    //)
    PRINT 31,Q11,Q22,Q33,Q44
 31 FORMAT (4F14.8)
    DO 88 K=1,250
    TIME (K)=K
 88 TIME(K)=T*TIME(K)
```

Cont.

```
C          THE NEXT GROUP OF CARDS EVALUATES THE OUTPUT OF THE
C      OBSERVABLE PROCESS.X1T AND X2T ARE THE REAL VALUES OF
C      DISTANCE AND SPEED,XNT IS THE MEASUREMENT NOISE AND Y IS
C      THE OUTPUT OF THE OBSERVABLE PROCESS.
       NUNIF=1220703125
       DO 105 I=1,IMAX1
       CALL RNDEV (NUNIF,DEV)
       XNT(I)=COE*DEV
       X1T(I)=X1T0       +R*SINF(ROT*(TIME(I)-TIME(1    )))
       X2T(I)=X2T0       *COSF(ROT*(TIME(I)-TIME(1    )))
  105  Y(I)=X1T(I)+XNT(I)
       DO 102 I=IMIN2,IMAX2
       CALL RNDEV (NUNIF,DEV)
       XNT(I)=COE*DEV
       X1T(I)=X1T(IMAX1)+(TIME(I)-TIME(IMAX1))*X2T(IMAX1)+
     2        0.5*(TIME(I)-TIME(IMAX1))**2*X3T02
       X2T(I)=X2T(IMAX1)+(TIME(I)-TIME(IMAX1))*X3T02
  102  Y(I)=X1T(I)+XNT(I)
       DO 103 I=IMIN3,IMAX3
       CALL RNDEV(NUNIF,DEV)
       XNT(I)=COE*DEV
       X1T(I)=X1T(IMAX2)+(TIME(I)-TIME(IMAX2))*X2T(IMAX2)+
     2        0.5*(TIME(I)-TIME(IMAX2))**2*X3T03
       X2T(I)=X2T(IMAX2)+(TIME(I)-TIME(IMAX2))*X3T03
  103  Y(I)=X1T(I)+XNT(I)
       DO 104 I=IMIN4,IMAX4
       CALL RNDEV (NUNIF,DEV)
       XNT(I)=COE*DEV
       X1T(I)=X1T(IMAX3)+(TIME(I)-TIME(IMAX3))*X2T(IMAX3)+
     2        0.5*(TIME(I)-TIME(IMAX3))**2*X3T04
       X2T(I)=X2T(IMAX3)+(TIME(I)-TIME(IMAX3))*X3T04
  104  Y(I)=X1T(I)+XNT(I)
       X1(3)=Y(3)
       X2(3)=(Y(3)-Y(2))/T
       X3(3)=(((Y(3)-Y(2))/T)-((Y(2)-Y(1))/T))/T
       XN(3)=0.0
       P11(3)=Q44
       P22(3)=2.0/(T**2)*Q44
       P33(3)=6.0/(T**4)*Q44
       P44(3)=Q44
       X1ES(2)=0.0
       VAX1ES(2)=0.0
       XNTS(2)=0.0
       VAXNTS(2)=0.0
       YERRS(2)=0.0
       VAYERRS(2)=0.0
  600  CONTINUE
       I=3
       J=I-1
       GO TO 10
   21  J=I-1                                              Cont.
```

```
C         THE NEXT GROUP OF CARDS EVALUATES THE COVARIANCE MATRIX.
      A11=(1.0-G1(J))
      A12=0.0
      A13=0.0
      A14=-G1(J)
      A21=-G2(J)
      A22=1.0
      A23=0.0
      A24=-G2(J)
      A31=-G3(J)
      A32=0.0
      A33=1.0
      A34=-G3(J)
      A41=-G4(J)
      A42=0.0
      A43=0.0
      A44=1.0-G4(J)
      B11=P11(J)+P12(J)*T+P13(J)*0.5*T**2
      B12=P12(J)+P13(J)*T
      B13=P13(J)
      B14=0.0
      B21=P21(J)+P22(J)*T+P23(J)*0.5*T**2
      B22=P22(J)+P23(J)*T
      B23=P23(J)
      B24=0.0
      B31=P31(J)+P32(J)*T+P33(J)*0.5*T**2
      B32=P32(J)+P33(J)*T
      B33=P33(J)
      B34=0.0
      B41=P41(J)+P42(J)*T+P43(J)*0.5*T**2
      B42=P42(J)+P43(J)*T
      B43=P43(J)
      B44=0.0
      C11=A11*B11+A14*B41
      C12=A11*B12+A14*B42
      C13=A11*B13+A14*B43
      C14=0.0
      C21=A21*B11+A22*B21+A24*B41
      C22=A21*B12+A22*B22+A24*B42
      C23=A21*B13+A22*B23+A24*B43
      C24=0.0
      C31=A31*B11+A33*B31+A34*B41
      C32=A31*B12+A33*B32+A34*B42
      C33=A31*B13+A33*B33+A34*B43
      C34=0.0
      C41=A41*B11+A44*B41
      C42=A41*B12+A44*B42
      C43=A41*B13+A44*B43
      C44=0.0
      P11(I)=C11+T*C21+0.5*T**2*C31+Q11
      P12(I)=C12+T*C22+0.5*T**2*C32
      P13(I)=C13+T*C23+0.5*T**2*C33
      P14(I)=0.0
      P21(I)=C21+T*C31
      P22(I)=C22+T*C32+Q22
      P23(I)=C23+T*C33
      P24(I)=0.0
      P31(I)=C31
      P32(I)=C32
      P33(I)=C33+Q33
      P34(I)=0.0
      P41(I)=0.0
      P42(I)=0.0
      P43(I)=0.0
      P44(I)=Q44
```

Cont.

```
C         THE NEXT GROUP OF CARDS EVALUATES THE GAINS AND THE
C      FILTERED STATE VECTOR.
   10 SIGMAS=P11(I)+P14(I)+P41(I)+P44(I)
      G1(I)=(P11(I)+P14(I))/SIGMAS
      G2(I)=(P21(I)+P24(I))/SIGMAS
      G3(I)=(P31(I)+P34(I))/SIGMAS
      G4(I)=(P41(I)+P44(I))/SIGMAS
      IF (I-3) 200,200,201
  200 YERROR(I)=Y(I)-(X1(I)+XN(I))
      X1(I)=X1(I)+YERROR(I)*G1(I)
      X2(I)=X2(I)+YERROR(I)*G2(I)
      X3(I)=X3(I)+YERROR(I)*G3(I)
      XN(I)=XN(I)+YERROR(I)*G4(I)
      GO TO 202
  201 YERROR(I)=Y(I)-(X1(J)+T*X2(J)+0.5*T**2*X3(J))
      X1(I)=X1(J)+T*X2(J)+0.5*T**2*X3(J)+YERROR(I)*G1(I)
      X2(I)=X2(J)+T*X3(J)+YERROR(I)*G2(I)
      X3(I)=X3(J)+YERROR(I)*G3(I)
      XN(I)=XN(J)+YERROR(I)*G4(I)
  202 X1E(I)=X1(I)-X1T(I)
      X2E(I)=X2(I)-X2T(I)
      TOTI=I-2
      X1ES(I)=X1ES(J)+X1E(I)
      AVX1E(I)=X1ES(I)/TOTI
      VAX1ES(I)=VAX1ES(J)+X1E(I)**2
      VAX1E(I)=VAX1ES(I)/TOTI
      XNTS(I)=XNTS(J)+XNT(I)
      AVXNT(I)=XNTS(I)/TOTI
      VAXNTS(I)=VAXNTS(J)+XNT(I)**2
      VAXNT(I)=VAXNTS(I)/TOTI
      YERRS(I)=YERRS(J)+YERROR(I)
      AVYERR(I)=YERRS(I)/TOTI
      VAYERRS(I)=VAYERRS(J)+YERROR(I)**2
      VAYERR(I)=VAYERRS(I)/TOTI
      SMOOTH(I)=((VAXNT(I)-VAX1E(I))/VAXNT(I))*100.0
      ABSXNT(I)=ABSF(XNT(I))+6.0
      ABSX1E(I)=-ABSF(X1E(I))+6.0
      I=I+1
      IF (I-IMAX4-1) 21,22,22
   22 CONTINUE
      PRINT 999
```

Cont.

```
C        THE NEXT GROUP OF CARDS PRINTS OUT THE RESULTS.
      PRINT 756
  756 FORMAT (//////61H
     2                Q33///)
      PRINT 757,Q33
  757 FORMAT(F68.15)
      PRINT 751
  751 FORMAT (///119H       TIME      XNT          X1T          X1         X1E
     2   X2T          X2          X2E           X3          AVXNT        AVX1E      SMCOT
     3H //)
      PRINT 750,((TIME(I),XNT(I),X1T(I),X1(I),X1E(I),X2T(I),X2(I),
     2            X2E(I),X3(I),AVXNT(I),AVX1E(I),SMOOTH(I)),I=4,IMAX4)
  750 FORMAT (F9.1,11E10.3)
      PRINT 999
      PRINT 752
  752 FORMAT (//////69H       TIME      P11         P22         P33         G1
     2       G2          G3      //)
      PRINT 753,((TIME(I),P11(I),P22(I),P33(I),G1(I),G2(I),G3(I)),
     2           I=4,IMAX4,2)
  753 FORMAT (F9.1,6E10.3)
      DO 11 L=1,12
   11 ITITLE(L)=8H
      ITITLE(1)=8HCARLOS P
      ITITLE(2)=8H. SIMOES
      ITITLE(4)=8H   MS  T
      ITITLE(5)=8HHESIS
      ITITLE(7)=8H      OP
      ITITLE(8)=8HTIMAL FI
      ITITLE(9)=8HLTERING
      ITITLE(10)=8HOF RADAR
      ITITLE(11)=8H DATA
      LABEL=4H P11
      CALL DRAW (149,TIME,P11,1,0,LABEL,ITITLE,0,0,0,0,0,0,5,6,1,LAST)
      LABEL=4H P22
      CALL DRAW (149,TIME,P22,2,0,LABEL)
      LABEL=4H P33
      CALL DRAW (149,TIME,P33,3,0,LABEL)
      LABEL=4HVXNT
      CALL DRAW (249,TIME,VAXNT,1,0,LABEL,ITITLE,0,0,0,0,0,0,5,6,1,LAST)
      LABEL=4HVX1E
      CALL DRAW (249,TIME,VAX1E,2,0,LABEL)
      LABEL=4H
      CALL DRAW (249,TIME,AVXNT,2,0,LABEL)
      LABEL=4H
      CALL DRAW (249,TIME,AVX1E,3,0,LABEL)
      CALL DRAW(249,TIME,ABSXNT,1,0,LABEL,ITITLE,0,0,0,0,0,0,5,6,1,LAST)
      LABEL=4HX1E
      CALL DRAW(249,TIME,ABSX1E,3,0,LABEL)
      LABEL=4HSMOO
      CALL DRAW(249,TIME,SMOOTH,0,0,LABEL,ITITLE,0,0,0,0,0,0,5,6,1,LAST)
  601 CONTINUE
      END
      END
```

INPUT DATA

COE

2.00

T

2.00

IMAX1	IMIN2	IMAX2	IMIN3	IMAX3	IMIN4	IMAX4
60	61	148	149	162	163	250

X1TO	X2TO	X3TO1	X3TO2	X3TO3	X3TO4
7.00000000	.27800000	.00000000	.00000000	-.00983000	.00000000

R	ROT
6.40000000	.05237000

Q11	Q22	Q33	Q44
.00000000	.00000000	.00000900	4.00000000

Case d)

OUTPUT DATA

TIME	XNT	X1T	X1	X1E	X2T
8.0	.263E+01	.898E+01	.106E+02	.166E+01	.264E+00
10.0	-.115E-01	.960E+01	.971E+01	.103E+00	.254E+00
12.0	-.158E+00	.102E+02	.976E+01	-.445E+00	.241E+00
14.0	.224E+01	.108E+02	.123E+02	.151E+01	.225E+00
16.0	-.247E+01	.113E+02	.989E+01	-.140E+01	.207E+00
18.0	-.190E+01	.118E+02	.958E+01	-.218E+01	.186E+00
20.0	-.119E+01	.122E+02	.102E+02	-.195E+01	.163E+00
22.0	-.209E+01	.125E+02	.102E+02	-.230E+01	.139E+00
24.0	-.136E+01	.128E+02	.109E+02	-.198E+01	.113E+00
26.0	-.150E+01	.131E+02	.113E+02	-.181E+01	.858E-01
28.0	.842E+00	.133E+02	.129E+02	-.406E+00	.577E-01
30.0	-.151E+01	.134E+02	.127E+02	-.687E+00	.290E-01
32.0	.394E+01	.134E+02	.151E+02	.174E+01	-.844E-04
34.0	-.335E+00	.134E+02	.148E+02	.141E+01	-.291E-01
36.0	.425E+01	.133E+02	.164E+02	.315E+01	-.579E-01
38.0	-.198E+01	.131E+02	.148E+02	.175E+01	-.860E-01
40.0	.816E+00	.128E+02	.147E+02	.181E+01	-.113E+00
42.0	.418E+01	.125E+02	.157E+02	.312E+01	-.139E+00
44.0	-.152E+01	.122E+02	.141E+02	.195E+01	-.163E+00
46.0	-.190E+01	.118E+02	.126E+02	.884E+00	-.186E+00
48.0	-.106E+01	.113E+02	.116E+02	.332E+00	-.207E+00
50.0	-.316E+00	.108E+02	.109E+02	.137E+00	-.225E+00
52.0	.215E+01	.102E+02	.110E+02	.783E+00	-.241E+00
54.0	-.761E+00	.960E+01	.993E+01	.327E+00	-.254E+00
56.0	-.926E+00	.897E+01	.886E+01	-.115E+00	-.264E+00
58.0	.215E+01	.833E+01	.880E+01	.478E+00	-.272E+00
60.0	-.951E+00	.767E+01	.764E+01	-.291E-01	-.276E+00
62.0	.850E+00	.700E+01	.709E+01	.924E-01	-.278E+00
64.0	.385E-01	.633E+01	.625E+01	-.765E-01	-.276E+00
66.0	.482E+00	.567E+01	.557E+01	-.977E-01	-.272E+00
68.0	-.131E+01	.502E+01	.434E+01	-.679E+00	-.264E+00
70.0	-.222E+01	.439E+01	.294E+01	-.145E+01	-.254E+00
72.0	.413E+00	.380E+01	.251E+01	-.129E+01	-.241E+00
74.0	-.191E+00	.323E+01	.191E+01	-.133E+01	-.225E+00
76.0	-.184E-01	.271E+01	.141E+01	-.131E+01	-.206E+00
78.0	-.624E+00	.224E+01	.778E+00	-.146E+01	-.186E+00
80.0	-.145E+01	.182E+01	-.171E-01	-.184E+01	-.163E+00
82.0	.700E+00	.145E+01	-.214E-01	-.148E+01	-.139E+00
84.0	-.145E+00	.115E+01	-.268E+00	-.142E+01	-.113E+00
86.0	-.171E+01	.912E+00	-.922E+00	-.183E+01	-.857E-01
88.0	.123E+01	.739E+00	-.527E+00	-.127E+01	-.576E-01
90.0	.262E+01	.634E+00	.286E+00	-.348E+00	-.288E-01
92.0	.215E+00	.600E+00	.289E+00	-.311E+00	.253E-03
94.0	.168E+01	.636E+00	.807E+00	.171E+00	.293E-01
96.0	.288E+01	.741E+00	.168E+01	.941E+00	.581E-01
98.0	.396E+01	.915E+00	.282E+01	.191E+01	.862E-01

X2	X2E	X3	AVXNT	AVX1E	SMOOTH
.332E+00	.673E-01	-.196E+00	.170E+01	.121E+01	.558E+02
-.703E+00	-.957E+00	-.360E+00	.113E+01	.839E+00	.557E+02
-.441E+00	-.682E+00	-.159E+00	.805E+00	.518E+00	.532E+02
.662E+00	.437E+00	.797E-01	.109E+01	.716E+00	.538E+02
-.443E+00	-.649E+00	-.981E-01	.499E+00	.364E+00	.584E+02
-.421E+00	-.607E+00	-.717E-01	.157E+00	.607E-03	.438E+02
-.155E+00	-.319E+00	-.287E-01	-.114E-01	-.243E+00	.311E+02
-.129E+00	-.268E+00	-.209E-01	-.243E+00	-.472E+00	.229E+02
.146E-01	-.984E-01	-.533E-02	-.355E+00	-.623E+00	.146E+02
.761E-01	-.975E-02	.165E-03	-.459E+00	-.730E+00	.104E+02
.314E+00	.256E+00	.167E-01	-.350E+00	-.703E+00	.118E+02
.218E+00	.189E+00	.840E-02	-.440E+00	-.702E+00	.162E+02
.523E+00	.523E+00	.258E-01	-.127E+00	-.527E+00	.359E+02
.379E+00	.408E+00	.147E-01	-.141E+00	-.398E+00	.321E+02
.515E+00	.573E+00	.204E-01	.134E+00	-.176E+00	.355E+02
.239E+00	.325E+00	.431E-02	.932E-02	-.630E-01	.348E+02
.173E+00	.286E+00	.653E-03	.542E-01	.413E-01	.309E+02
.246E+00	.385E+00	.402E-02	.271E+00	.203E+00	.335E+02
.408E-01	.204E+00	-.572E-02	.182E+00	.291E+00	.311E+02
-.128E+00	.585E-01	-.127E-01	.826E-01	.319E+00	.328E+02
-.226E+00	-.194E-01	-.160E-01	.305E-01	.320E+00	.335E+02
-.280E+00	-.552E-01	-.169E-01	.154E-01	.312E+00	.335E+02
-.250E+00	-.957E-02	-.142E-01	.104E+00	.331E+00	.359E+02
-.327E+00	-.732E-01	-.163E-01	.698E-01	.331E+00	.362E+02
-.395E+00	-.130E+00	-.178E-01	.315E-01	.314E+00	.367E+02
-.360E+00	-.882E-01	-.147E-01	.110E+00	.320E+00	.392E+02
-.428E+00	-.151E+00	-.164E-01	.721E-01	.308E+00	.397E+02
-.430E+00	-.152E+00	-.150E-01	.989E-01	.300E+00	.401E+02
-.455E+00	-.179E+00	-.148E-01	.969E-01	.288E+00	.401E+02
-.461E+00	-.190E+00	-.138E-01	.109E+00	.275E+00	.402E+02
-.514E+00	-.250E+00	-.149E-01	.650E-01	.245E+00	.407E+02
-.575E+00	-.321E+00	-.163E-01	-.418E-02	.194E+00	.414E+02
-.539E+00	-.299E+00	-.132E-01	.809E-02	.150E+00	.401E+02
-.520E+00	-.295E+00	-.112E-01	.241E-02	.108E+00	.386E+02
-.491E+00	-.285E+00	-.883E-02	.183E-02	.688E-01	.371E+02
-.475E+00	-.289E+00	-.731E-02	-.151E-01	.275E-01	.355E+02
-.474E+00	-.311E+00	-.661E-02	-.528E-01	-.216E-01	.338E+02
-.401E+00	-.262E+00	-.267E-02	-.335E-01	-.589E-01	.323E+02
-.355E+00	-.242E+00	-.368E-03	-.363E-01	-.929E-01	.306E+02
-.351E+00	-.265E+00	-.138E-03	-.771E-01	-.135E+00	.295E+02
-.251E+00	-.194E+00	.437E-02	-.459E-01	-.162E+00	.291E+02
-.124E+00	-.953E-01	.973E-02	.162E-01	-.167E+00	.327E+02
-.836E-01	-.839E-01	.107E-01	.207E-01	-.170E+00	.327E+02
-.207E-02	-.314E-01	.134E-01	.577E-01	-.162E+00	.341E+02
.102E+00	.439E-01	.169E-01	.119E+00	-.138E+00	.373E+02
.217E+00	.131E+00	.206E-01	.201E+00	-.948E-01	.412E+02

TIME	XNT	X1T	X1	X1E	X2T
100.0	.274E+01	.116E+01	.348E+01	.232E+01	.113E+00
102.0	.521E+01	.146E+01	.486E+01	.340E+01	.139E+00
104.0	-.403E+01	.183E+01	.329E+01	.146E+01	.164E+00
106.0	.253E+00	.225E+01	.337E+01	.113E+01	.186E+00
108.0	.205E+00	.272E+01	.356E+01	.834E+00	.207E+00
110.0	.171E+01	.324E+01	.430E+01	.106E+01	.225E+00
112.0	-.301E+01	.381E+01	.362E+01	-.187E+00	.241E+00
114.0	-.192E+01	.440E+01	.354E+01	-.868E+00	.254E+00
116.0	-.270E+01	.503E+01	.339E+01	-.164E+01	.264E+00
118.0	-.150E+01	.568E+01	.381E+01	-.187E+01	.272E+00
120.0	-.343E+00	.634E+01	.469E+01	-.165E+01	.277E+00
122.0	-.228E+01	.689E+01	.497E+01	-.192E+01	.277E+00
124.0	-.154E+01	.744E+01	.556E+01	-.189E+01	.277E+00
126.0	.856E-01	.800E+01	.667E+01	-.133E+01	.277E+00
128.0	-.772E+00	.855E+01	.745E+01	-.110E+01	.277E+00
130.0	-.163E+01	.910E+01	.795E+01	-.116E+01	.277E+00
132.0	.982E+00	.966E+01	.926E+01	-.400E+00	.277E+00
134.0	-.398E+01	.102E+02	.893E+01	-.128E+01	.277E+00
136.0	-.632E+00	.108E+02	.977E+01	-.992E+00	.277E+00
138.0	-.152E+00	.113E+02	.107E+02	-.605E+00	.277E+00
140.0	-.154E+01	.119E+02	.112E+02	-.710E+00	.277E+00
142.0	-.101E+01	.124E+02	.118E+02	-.648E+00	.277E+00
144.0	-.168E+01	.130E+02	.122E+02	-.808E+00	.277E+00
146.0	.434E+01	.135E+02	.144E+02	.884E+00	.277E+00
148.0	-.188E+01	.141E+02	.145E+02	.377E+00	.277E+00
150.0	.155E+01	.146E+02	.156E+02	.969E+00	.277E+00
152.0	.103E+01	.152E+02	.165E+02	.128E+01	.277E+00
154.0	.256E+01	.157E+02	.177E+02	.197E+01	.277E+00
156.0	-.171E+01	.163E+02	.175E+02	.120E+01	.277E+00
158.0	.349E+00	.168E+02	.180E+02	.115E+01	.277E+00
160.0	-.166E+01	.174E+02	.179E+02	.468E+00	.277E+00
162.0	-.203E+01	.180E+02	.177E+02	-.237E+00	.277E+00
164.0	.532E+00	.185E+02	.185E+02	-.541E-01	.277E+00
166.0	.134E+01	.191E+02	.194E+02	.339E+00	.277E+00
168.0	-.499E-01	.196E+02	.198E+02	.236E+00	.277E+00
170.0	-.144E+00	.202E+02	.203E+02	.113E+00	.277E+00
172.0	-.247E+01	.207E+02	.200E+02	-.704E+00	.277E+00
174.0	-.449E+01	.213E+02	.193E+02	-.200E+01	.277E+00
176.0	.143E+01	.218E+02	.206E+02	-.125E+01	.277E+00
178.0	.257E+01	.224E+02	.221E+02	-.264E+00	.277E+00
180.0	.250E+01	.229E+02	.235E+02	.551E+00	.277E+00
182.0	-.472E+01	.235E+02	.225E+02	-.974E+00	.277E+00
184.0	.162E+01	.240E+02	.237E+02	-.309E+00	.277E+00
186.0	.109E+01	.246E+02	.247E+02	.893E-01	.277E+00
188.0	-.983E+00	.251E+02	.249E+02	-.208E+00	.277E+00
190.0	-.162E+00	.257E+02	.255E+02	-.206E+00	.277E+00
192.0	.847E+00	.262E+02	.264E+02	.105E+00	.277E+00
194.0	-.378E+01	.268E+02	.258E+02	-.105E+01	.277E+00
196.0	-.163E+00	.274E+02	.265E+02	-.899E+00	.277E+00
198.0	.209E+01	.279E+02	.278E+02	-.776E-01	.277E+00

X2	X2E	X3	AVXNT	AVX1E	SMOOTH
.275E+00	.162E+00	.213E-01	.254E+00	-.444E-01	.406E+02
.390E+00	.250E+00	.246E-01	.355E+00	.259E-01	.430E+02
.221E+00	.572E-01	.147E-01	.267E+00	.546E-01	.463E+02
.216E+00	.293E-01	.132E-01	.267E+00	.756E-01	.458E+02
.217E+00	.101E-01	.120E-01	.266E+00	.902E-01	.454E+02
.267E+00	.417E-01	.132E-01	.293E+00	.108E+00	.457E+02
.181E+00	-.601E-01	.810E-02	.232E+00	.103E+00	.479E+02
.155E+00	-.989E-01	.621E-02	.192E+00	.853E-01	.484E+02
.125E+00	-.139E+00	.429E-02	.141E+00	.545E-01	.489E+02
.148E+00	-.124E+00	.494E-02	.112E+00	.208E-01	.479E+02
.210E+00	-.664E-01	.729E-02	.104E+00	-.795E-02	.467E+02
.211E+00	-.659E-01	.666E-02	.637E-01	-.404E-01	.463E+02
.238E+00	-.388E-01	.728E-02	.370E-01	-.712E-01	.454E+02
.309E+00	.320E-01	.983E-02	.378E-01	-.919E-01	.447E+02
.341E+00	.648E-01	.104E-01	.247E-01	-.108E+00	.443E+02
.343E+00	.670E-01	.958E-02	-.152E-02	-.125E+00	.444E+02
.418E+00	.141E+00	.121E-01	.139E-01	-.129E+00	.445E+02
.334E+00	.577E-01	.720E-02	-.477E-01	-.147E+00	.472E+02
.363E+00	.864E-01	.784E-02	-.565E-01	-.160E+00	.470E+02
.397E+00	.120E+00	.866E-02	-.580E-01	-.166E+00	.468E+02
.381E+00	.104E+00	.717E-02	-.797E-01	-.174E+00	.471E+02
.381E+00	.104E+00	.651E-02	-.933E-01	-.181E+00	.472E+02
.359E+00	.827E-01	.494E-02	-.116E+00	-.190E+00	.475E+02
.506E+00	.230E+00	.112E-01	-.531E-01	-.175E+00	.506E+02
.439E+00	.163E+00	.711E-02	-.785E-01	-.167E+00	.512E+02
.476E+00	.200E+00	.815E-02	-.562E-01	-.152E+00	.513E+02
.483E+00	.206E+00	.770E-02	-.416E-01	-.132E+00	.509E+02
.522E+00	.245E+00	.877E-02	-.685E-02	-.104E+00	.507E+02
.424E+00	.147E+00	.354E-02	-.293E-01	-.873E-01	.507E+02
.399E+00	.123E+00	.210E-02	-.244E-01	-.713E-01	.503E+02
.319E+00	.425E-01	-.172E-02	-.453E-01	-.643E-01	.506E+02
.244E+00	-.322E-01	-.495E-02	-.704E-01	-.665E-01	.513E+02
.258E+00	-.188E-01	-.390E-02	-.629E-01	-.664E-01	.513E+02
.290E+00	.131E-01	-.210E-02	-.456E-01	-.614E-01	.516E+02
.274E+00	-.246E-02	-.261E-02	-.456E-01	-.577E-01	.515E+02
.259E+00	-.179E-01	-.307E-02	-.468E-01	-.557E-01	.515E+02
.182E+00	-.942E-01	-.625E-02	-.757E-01	-.634E-01	.523E+02
.708E-01	-.206E+00	-.107E-01	-.128E+00	-.862E-01	.540E+02
.156E+00	-.120E+00	-.590E-02	-.109E+00	-.997E-01	.538E+02
.257E+00	-.198E-01	-.808E-03	-.787E-01	-.102E+00	.546E+02
.333E+00	.561E-01	.271E-02	-.493E-01	-.942E-01	.553E+02
.189E+00	-.873E-01	-.404E-02	-.102E+00	-.104E+00	.577E+02
.258E+00	-.186E-01	-.557E-03	-.826E-01	-.106E+00	.580E+02
.296E+00	.199E-01	.123E-02	-.698E-01	-.104E+00	.581E+02
.268E+00	-.841E-02	-.159E-03	-.797E-01	-.105E+00	.582E+02
.270E+00	-.698E-02	-.798E-04	-.806E-01	-.106E+00	.582E+02
.299E+00	.223E-01	.125E-02	-.707E-01	-.104E+00	.583E+02
.193E+00	-.838E-01	-.367E-02	-.110E+00	-.114E+00	.595E+02
.215E+00	-.619E-01	-.234E-02	-.110E+00	-.122E+00	.593E+02
.296E+00	.195E-01	.156E-02	-.877E-01	-.122E+00	.598E+02

TIME	XNT	X1T	X1	X1E	X2T
200.0	-.801E+00	.285E+02	.282E+02	-.269E+00	.277E+00
202.0	-.194E+00	.290E+02	.288E+02	-.243E+00	.277E+00
204.0	.181E+00	.296E+02	.295E+02	-.106E+00	.277E+00
206.0	-.269E+01	.301E+02	.292E+02	-.869E+00	.277E+00
208.0	.174E+01	.307E+02	.305E+02	-.149E+00	.277E+00
210.0	-.166E+01	.312E+02	.306E+02	-.581E+00	.277E+00
212.0	.197E+01	.318E+02	.319E+02	.165E+00	.277E+00
214.0	.217E+01	.323E+02	.332E+02	.849E+00	.277E+00
216.0	.119E+01	.329E+02	.340E+02	.111E+01	.277E+00
218.0	-.252E+00	.334E+02	.343E+02	.869E+00	.277E+00
220.0	-.407E-01	.340E+02	.347E+02	.715E+00	.277E+00
222.0	-.618E+00	.345E+02	.349E+02	.398E+00	.277E+00
224.0	.417E+00	.351E+02	.355E+02	.439E+00	.277E+00
226.0	.218E+01	.356E+02	.367E+02	.100E+01	.277E+00
228.0	.254E+01	.362E+02	.378E+02	.157E+01	.277E+00
230.0	-.357E-01	.368E+02	.380E+02	.125E+01	.277E+00
232.0	.324E+00	.373E+02	.384E+02	.106E+01	.277E+00
234.0	-.771E+00	.379E+02	.384E+02	.566E+00	.277E+00
236.0	-.293E+01	.384E+02	.379E+02	-.513E+00	.277E+00
238.0	.226E-02	.390E+02	.384E+02	-.518E+00	.277E+00
240.0	-.137E+00	.395E+02	.390E+02	-.551E+00	.277E+00
242.0	-.274E+01	.401E+02	.387E+02	-.136E+01	.277E+00
244.0	.157E+00	.406E+02	.395E+02	-.113E+01	.277E+00
246.0	.314E+01	.412E+02	.412E+02	-.977E-02	.277E+00
248.0	.295E+01	.417E+02	.426E+02	.878E+00	.277E+00
250.0	-.186E+00	.423E+02	.429E+02	.662E+00	.277E+00
252.0	.427E+00	.428E+02	.435E+02	.661E+00	.277E+00
254.0	.154E+01	.434E+02	.444E+02	.995E+00	.277E+00
256.0	.113E+01	.439E+02	.451E+02	.114E+01	.277E+00
258.0	.653E+00	.445E+02	.456E+02	.110E+01	.277E+00
260.0	.155E+00	.451E+02	.460E+02	.910E+00	.277E+00
262.0	.120E+00	.456E+02	.463E+02	.729E+00	.277E+00
264.0	.307E+01	.462E+02	.476E+02	.147E+01	.277E+00
266.0	-.613E+00	.467E+02	.477E+02	.955E+00	.277E+00
268.0	-.354E+00	.473E+02	.479E+02	.594E+00	.277E+00
270.0	-.250E+01	.478E+02	.475E+02	-.365E+00	.277E+00
272.0	.160E+01	.484E+02	.485E+02	.885E-01	.277E+00
274.0	.146E+01	.489E+02	.494E+02	.433E+00	.277E+00
276.0	.712E+00	.495E+02	.500E+02	.497E+00	.277E+00
278.0	-.185E+01	.500E+02	.498E+02	-.230E+00	.277E+00
280.0	-.301E+00	.506E+02	.502E+02	-.357E+00	.277E+00
282.0	.254E+01	.511E+02	.515E+02	.415E+00	.277E+00
284.0	-.373E+01	.517E+02	.508E+02	-.846E+00	.277E+00
286.0	-.151E+01	.522E+02	.510E+02	-.121E+01	.277E+00
288.0	.142E+00	.528E+02	.518E+02	-.993E+00	.277E+00
290.0	.700E+00	.533E+02	.527E+02	-.620E+00	.277E+00
292.0	.326E+01	.539E+02	.544E+02	.488E+00	.277E+00
294.0	-.944E+00	.545E+02	.546E+02	.134E+00	.277E+00
296.0	.262E+00	.550E+02	.552E+02	.206E+00	.277E+00
298.0	-.192E+01	.555E+02	.552E+02	-.387E+00	.257E+00

X2	X2E	X3	AVXNT	AVX1E	SMOOTH
.278E+00	.145E-02	.599E-03	-.950E-01	-.123E+00	.598E+02
.281E+00	.460E-02	.688E-03	-.960E-01	-.125E+00	.598E+02
.294E+00	.174E-01	.121E-02	-.932E-01	-.124E+00	.598E+02
.224E+00	-.527E-01	-.208E-02	-.119E+00	-.132E+00	.603E+02
.295E+00	.184E-01	.133E-02	-.101E+00	-.132E+00	.606E+02
.255E+00	-.217E-01	-.606E-03	-.116E+00	-.136E+00	.608E+02
.325E+00	.487E-01	.264E-02	-.957E-01	-.133E+00	.611E+02
.383E+00	.107E+00	.502E-02	-.741E-01	-.124E+00	.614E+02
.397E+00	.120E+00	.517E-02	-.621E-01	-.112E+00	.612E+02
.362E+00	.859E-01	.316E-02	-.639E-01	-.103E+00	.611E+02
.339E+00	.622E-01	.180E-02	-.637E-01	-.956E-01	.610E+02
.302E+00	.255E-01	-.284E-04	-.688E-01	-.911E-01	.610E+02
.301E+00	.245E-01	-.686E-04	-.644E-01	-.862E-01	.609E+02
.348E+00	.710E-01	.204E-02	-.442E-01	-.764E-01	.611E+02
.390E+00	.113E+00	.378E-02	-.211E-01	-.617E-01	.611E+02
.347E+00	.701E-01	.148E-02	-.213E-01	-.501E-01	.608E+02
.320E+00	.437E-01	.146E-03	-.182E-01	-.404E-01	.605E+02
.267E+00	-.908E-02	-.226E-02	-.248E-01	-.351E-01	.605E+02
.167E+00	-.109E+00	-.660E-02	-.498E-01	-.392E-01	.612E+02
.175E+00	-.102E+00	-.566E-02	-.493E-01	-.433E-01	.611E+02
.180E+00	-.968E-01	-.492E-02	-.501E-01	-.476E-01	.611E+02
.115E+00	-.161E+00	-.739E-02	-.726E-01	-.586E-01	.613E+02
.152E+00	-.125E+00	-.508E-02	-.707E-01	-.676E-01	.610E+02
.267E+00	-.994E-02	.593E-03	-.442E-01	-.671E-01	.619E+02
.350E+00	.734E-01	.431E-02	-.197E-01	-.593E-01	.624E+02
.325E+00	.483E-01	.279E-02	-.210E-01	-.535E-01	.623E+02
.321E+00	.446E-01	.237E-02	-.174E-01	-.477E-01	.622E+02
.348E+00	.711E-01	.335E-02	-.492E-02	-.394E-01	.622E+02
.354E+00	.775E-01	.334E-02	.411E-02	-.300E-01	.620E+02
.343E+00	.662E-01	.252E-02	.922E-02	-.211E-01	.618E+02
.318E+00	.412E-01	.117E-02	.104E-01	-.138E-01	.616E+02
.296E+00	.194E-01	.707E-04	.112E-01	-.805E-02	.615E+02
.360E+00	.832E-01	.295E-02	.348E-01	.332E-02	.618E+02
.303E+00	.268E-01	.133E-03	.298E-01	.106E-01	.617E+02
.266E+00	-.105E-01	-.157E-02	.269E-01	.150E-01	.616E+02
.178E+00	-.983E-01	-.541E-02	.792E-02	.121E-01	.621E+02
.227E+00	-.493E-01	-.269E-02	.198E-01	.127E-01	.623E+02
.263E+00	-.140E-01	-.853E-03	.304E-01	.158E-01	.624E+02
.269E+00	-.719E-02	-.467E-03	.354E-01	.194E-01	.624E+02
.204E+00	-.726E-01	-.339E-02	.216E-01	.176E-01	.626E+02
.199E+00	-.772E-01	-.329E-02	.193E-01	.148E-01	.626E+02
.277E+00	.792E-03	.542E-03	.375E-01	.177E-01	.631E+02
.164E+00	-.113E+00	-.465E-02	.106E-01	.115E-01	.639E+02
.143E+00	-.134E+00	-.519E-02	-.251E-03	.286E-02	.638E+02
.177E+00	-.992E-01	-.315E-02	.748E-03	-.415E-02	.636E+02
.223E+00	-.531E-01	-.771E-03	.564E-02	-.846E-02	.636E+02
.332E+00	.555E-01	.422E-02	.282E-01	-.501E-02	.642E+02
.298E+00	.211E-01	.228E-02	.215E-01	-.405E-02	.643E+02
.304E+00	.279E-01	.238E-02	.232E-01	-.261E-02	.643E+02
.248E+00	-.872E-02	-.386E-03	.994E-02	-.523E-02	.645E+02

TIME	XNT	X1T	X1	X1E	X2T
300.0	-.145E+01	.560E+02	.553E+02	-.710E+00	.237E+00
302.0	-.706E+00	.565E+02	.558E+02	-.725E+00	.218E+00
304.0	.632E+00	.569E+02	.566E+02	-.303E+00	.198E+00
306.0	-.808E+00	.573E+02	.569E+02	-.372E+00	.178E+00
308.0	-.138E+01	.576E+02	.570E+02	-.593E+00	.159E+00
310.0	.101E+01	.579E+02	.579E+02	-.398E-01	.139E+00
312.0	-.141E+00	.582E+02	.582E+02	.775E-01	.119E+00
314.0	.540E+00	.584E+02	.588E+02	.380E+00	.996E-01
316.0	.658E+00	.586E+02	.592E+02	.661E+00	.799E-01
318.0	.144E+01	.587E+02	.598E+02	.112E+01	.603E-01
320.0	-.272E+00	.588E+02	.598E+02	.981E+00	.406E-01
322.0	.985E+00	.589E+02	.601E+02	.121E+01	.209E-01
324.0	-.937E-01	.589E+02	.599E+02	.105E+01	.127E-02
326.0	-.153E+01	.589E+02	.593E+02	.436E+00	.127E-02
328.0	.203E+01	.589E+02	.599E+02	.971E+00	.127E-02
330.0	.176E+01	.589E+02	.602E+02	.130E+01	.127E-02
332.0	.349E+01	.589E+02	.610E+02	.206E+01	.127E-02
334.0	-.245E+01	.589E+02	.598E+02	.853E+00	.127E-02
336.0	-.688E+00	.589E+02	.593E+02	.350E+00	.127E-02
338.0	.126E+01	.589E+02	.594E+02	.510E+00	.127E-02
340.0	-.248E+01	.589E+02	.584E+02	-.506E+00	.127E-02
342.0	-.443E+00	.589E+02	.582E+02	-.729E+00	.127E-02
344.0	.144E+01	.589E+02	.586E+02	-.330E+00	.127E-02
346.0	.260E+00	.589E+02	.586E+02	-.344E+00	.127E-02
348.0	.780E+00	.589E+02	.587E+02	-.185E+00	.127E-02
350.0	.429E+01	.589E+02	.600E+02	.103E+01	.127E-02
352.0	.685E+00	.589E+02	.599E+02	.945E+00	.127E-02
354.0	-.330E+01	.589E+02	.586E+02	-.343E+00	.127E-02
356.0	-.106E+01	.589E+02	.582E+02	-.731E+00	.127E-02
358.0	-.168E+01	.589E+02	.577E+02	-.123E+01	.127E-02
360.0	.120E+01	.589E+02	.582E+02	-.736E+00	.127E-02
362.0	-.436E+00	.589E+02	.581E+02	-.804E+00	.127E-02
364.0	.338E+01	.589E+02	.593E+02	.323E+00	.127E-02
366.0	.175E+01	.589E+02	.597E+02	.776E+00	.127E-02
368.0	-.175E+01	.589E+02	.590E+02	.857E-01	.127E-02
370.0	-.234E+01	.590E+02	.583E+02	-.669E+00	.127E-02
372.0	.105E+01	.590E+02	.587E+02	-.255E+00	.127E-02
374.0	-.104E+01	.590E+02	.584E+02	-.536E+00	.127E-02
376.0	.728E+00	.590E+02	.587E+02	-.222E+00	.127E-02
378.0	-.223E+01	.590E+02	.581E+02	-.853E+00	.127E-02
380.0	-.587E+00	.590E+02	.581E+02	-.869E+00	.127E-02
382.0	-.779E+00	.590E+02	.580E+02	-.927E+00	.127E-02
384.0	-.197E-01	.590E+02	.582E+02	-.731E+00	.127E-02
386.0	-.135E+01	.590E+02	.580E+02	-.962E+00	.127E-02
388.0	.226E+01	.590E+02	.589E+02	-.452E-01	.127E-02
390.0	-.317E+01	.590E+02	.580E+02	-.928E+00	.127E-02
392.0	.974E+00	.590E+02	.586E+02	-.400E+00	.127E-02
394.0	.142E+01	.590E+02	.592E+02	.182E+00	.127E-02
396.0	.275E+01	.590E+02	.601E+02	.107E+01	.127E-02
398.0	.604E+00	.590E+02	.601E+02	.114E+01	.127E-02

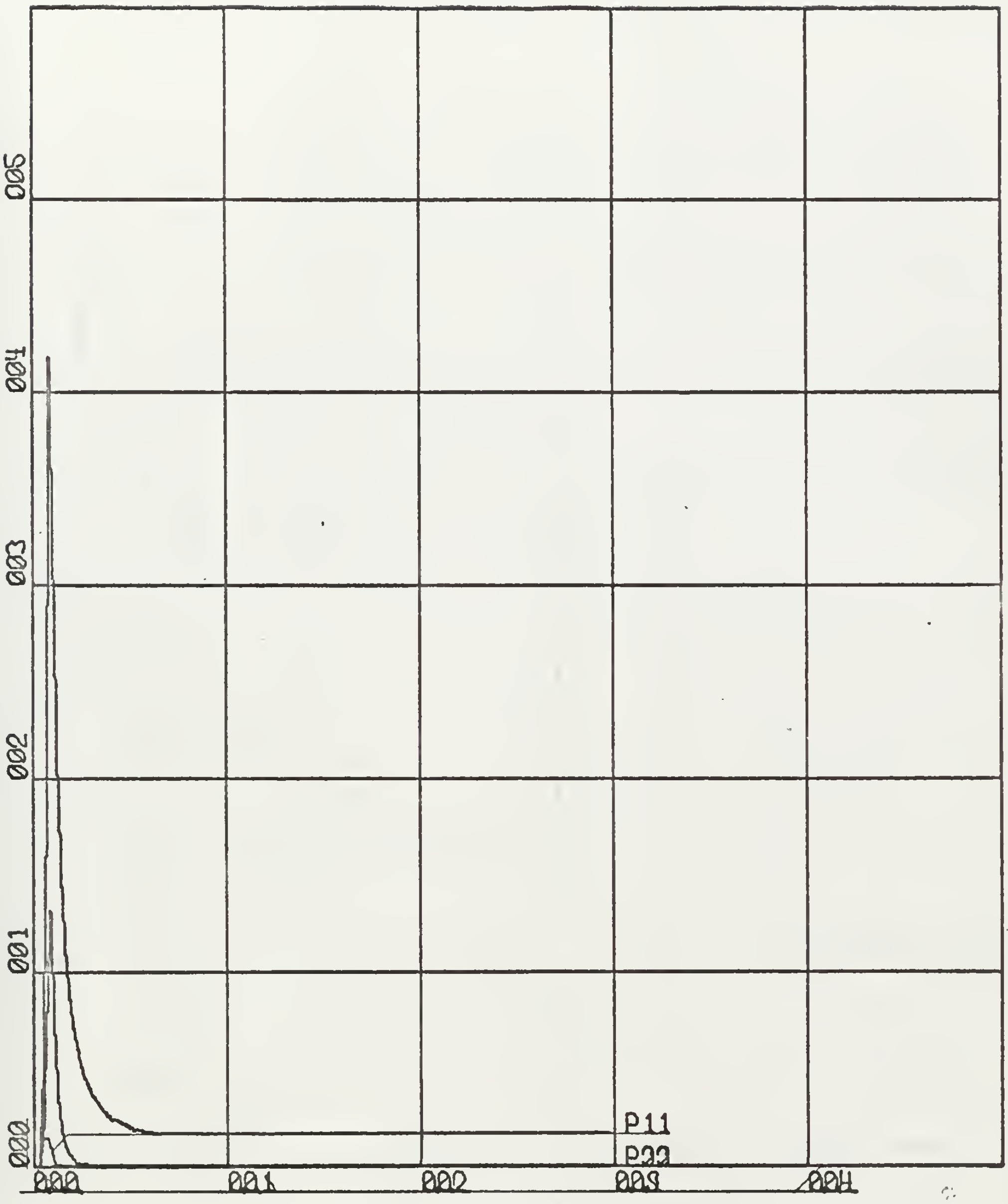
X2	X2E	X3	AVXNT	AVX1E	SMOOTH
.218E+00	-.192E-01	-.172E-02	.651E-04	-.999E-02	.645E+02
.215E+00	-.229E-02	-.169E-02	-.467E-02	-.148E-01	.645E+02
.249E+00	.511E-01	-.362E-05	-.432E-03	-.167E-01	.645E+02
.232E+00	.535E-01	-.788E-03	-.578E-02	-.191E-01	.645E+02
.199E+00	.402E-01	-.221E-02	-.148E-01	-.228E-01	.646E+02
.236E+00	.969E-01	-.329E-03	-.818E-02	-.230E-01	.646E+02
.227E+00	.107E+00	-.721E-03	-.904E-02	-.223E-01	.646E+02
.231E+00	.132E+00	-.434E-03	-.550E-02	-.197E-01	.646E+02
.230E+00	.151E+00	-.439E-03	-.124E-02	-.153E-01	.646E+02
.242E+00	.182E+00	.129E-03	.791E-02	-.811E-02	.645E+02
.195E+00	.154E+00	-.202E-02	.652E-02	-.185E-02	.643E+02
.182E+00	.161E+00	-.242E-02	.127E-01	.575E-02	.641E+02
.132E+00	.131E+00	-.447E-02	.120E-01	.122E-01	.639E+02
.450E-01	.437E-01	-.801E-02	.242E-02	.149E-01	.640E+02
.710E-01	.698E-01	-.610E-02	.149E-01	.208E-01	.641E+02
.770E-01	.757E-01	-.528E-02	.256E-01	.286E-01	.640E+02
.123E+00	.122E+00	-.272E-02	.467E-01	.410E-01	.640E+02
-.138E-01	-.150E-01	-.866E-02	.316E-01	.460E-01	.643E+02
-.723E-01	-.736E-01	-.105E-01	.272E-01	.478E-01	.643E+02
-.637E-01	-.650E-01	-.919E-02	.346E-01	.506E-01	.643E+02
-.161E+00	-.162E+00	-.127E-01	.196E-01	.472E-01	.647E+02
-.175E+00	-.176E+00	-.122E-01	.169E-01	.426E-01	.646E+02
-.129E+00	-.130E+00	-.904E-02	.253E-01	.405E-01	.647E+02
-.123E+00	-.124E+00	-.796E-02	.266E-01	.382E-01	.647E+02
-.101E+00	-.102E+00	-.622E-02	.310E-01	.369E-01	.647E+02
.165E-01	.152E-01	-.354E-03	.556E-01	.426E-01	.656E+02
.550E-02	.422E-02	-.821E-03	.593E-01	.478E-01	.655E+02
-.113E+00	-.115E+00	-.613E-02	.401E-01	.456E-01	.661E+02
-.139E+00	-.140E+00	-.673E-02	.338E-01	.412E-01	.661E+02
-.171E+00	-.172E+00	-.756E-02	.241E-01	.340E-01	.660E+02
-.109E+00	-.110E+00	-.408E-02	.307E-01	.297E-01	.660E+02
-.102E+00	-.104E+00	-.342E-02	.281E-01	.251E-01	.659E+02
.123E-01	.110E-01	.209E-02	.467E-01	.267E-01	.665E+02
.550E-01	.538E-01	.384E-02	.561E-01	.308E-01	.665E+02
-.102E-01	-.115E-01	.533E-03	.462E-01	.311E-01	.667E+02
-.756E-01	-.768E-01	-.247E-02	.332E-01	.273E-01	.669E+02
-.287E-01	-.299E-01	-.126E-03	.387E-01	.258E-01	.670E+02
-.489E-01	-.501E-01	-.103E-02	.329E-01	.228E-01	.670E+02
-.132E-01	-.145E-01	.680E-03	.366E-01	.214E-01	.670E+02
-.666E-01	-.678E-01	-.180E-02	.245E-01	.168E-01	.671E+02
-.589E-01	-.602E-01	-.129E-02	.212E-01	.120E-01	.670E+02
-.557E-01	-.569E-01	-.103E-02	.170E-01	.708E-02	.669E+02
-.295E-01	-.307E-01	.254E-03	.168E-01	.320E-02	.668E+02
-.444E-01	-.457E-01	-.445E-03	.964E-02	-.186E-02	.668E+02
.462E-01	.449E-01	.370E-02	.214E-01	-.208E-02	.670E+02
-.353E-01	-.365E-01	-.326E-03	.484E-02	-.688E-02	.674E+02
.186E-01	.174E-01	.215E-02	.984E-02	-.891E-02	.674E+02
.721E-01	.708E-01	.437E-02	.171E-01	-.793E-02	.675E+02
.148E+00	.146E+00	.739E-02	.310E-01	-.244E-02	.677E+02
.141E+00	.140E+00	.643E-02	.339E-01	.336E-02	.675E+02

TIME	XNT	X1T	X1	X1E	X2T
400.0	-.195E+01	.590E+02	.594E+02	.403E+00	.127E-02
402.0	.265E+01	.590E+02	.602E+02	.117E+01	.127E-02
404.0	-.991E+00	.590E+02	.597E+02	.690E+00	.127E-02
406.0	-.255E+01	.590E+02	.588E+02	-.204E+00	.127E-02
408.0	.266E+01	.590E+02	.596E+02	.632E+00	.127E-02
410.0	.126E+01	.590E+02	.599E+02	.896E+00	.127E-02
412.0	-.413E+01	.590E+02	.585E+02	-.537E+00	.127E-02
414.0	.911E+00	.590E+02	.588E+02	-.200E+00	.127E-02
416.0	.175E+01	.590E+02	.594E+02	.341E+00	.127E-02
418.0	.284E+00	.590E+02	.594E+02	.346E+00	.127E-02
420.0	.139E+01	.590E+02	.597E+02	.683E+00	.127E-02
422.0	.264E+01	.590E+02	.604E+02	.134E+01	.127E-02
424.0	.110E+01	.590E+02	.604E+02	.140E+01	.127E-02
426.0	.166E+01	.590E+02	.606E+02	.161E+01	.127E-02
428.0	-.842E+00	.590E+02	.600E+02	.100E+01	.127E-02
430.0	-.159E+01	.590E+02	.593E+02	.255E+00	.127E-02
432.0	.108E+01	.590E+02	.595E+02	.440E+00	.127E-02
434.0	.103E+01	.590E+02	.596E+02	.581E+00	.127E-02
436.0	.231E+00	.590E+02	.595E+02	.455E+00	.127E-02
438.0	-.328E+01	.590E+02	.583E+02	-.721E+00	.127E-02
440.0	.580E-01	.590E+02	.584E+02	-.674E+00	.127E-02
442.0	-.136E+00	.590E+02	.584E+02	-.677E+00	.127E-02
444.0	-.142E+00	.590E+02	.584E+02	-.665E+00	.127E-02
446.0	-.203E+00	.590E+02	.584E+02	-.656E+00	.127E-02
448.0	.141E+01	.591E+02	.589E+02	-.142E+00	.127E-02
450.0	-.319E+01	.591E+02	.579E+02	-.110E+01	.127E-02
452.0	-.159E-01	.591E+02	.581E+02	-.926E+00	.127E-02
454.0	-.334E+01	.591E+02	.573E+02	-.177E+01	.127E-02
456.0	.133E+01	.591E+02	.580E+02	-.103E+01	.127E-02
458.0	-.186E+00	.591E+02	.582E+02	-.859E+00	.127E-02
460.0	.591E+00	.591E+02	.586E+02	-.463E+00	.127E-02
462.0	-.551E+00	.591E+02	.586E+02	-.473E+00	.127E-02
464.0	-.739E+00	.591E+02	.585E+02	-.535E+00	.127E-02
466.0	.130E+00	.591E+02	.588E+02	-.318E+00	.127E-02
468.0	.143E+01	.591E+02	.593E+02	.262E+00	.127E-02
470.0	.186E+00	.591E+02	.594E+02	.360E+00	.127E-02
472.0	-.214E+01	.591E+02	.588E+02	-.275E+00	.127E-02
474.0	.178E+01	.591E+02	.595E+02	.381E+00	.127E-02
476.0	-.490E+01	.591E+02	.580E+02	-.111E+01	.127E-02
478.0	.190E+01	.591E+02	.588E+02	-.286E+00	.127E-02
480.0	-.426E+00	.591E+02	.588E+02	-.303E+00	.127E-02
482.0	.248E+01	.591E+02	.597E+02	.567E+00	.127E-02
484.0	-.179E+01	.591E+02	.591E+02	-.171E-01	.127E-02
486.0	-.383E+00	.591E+02	.590E+02	-.855E-01	.127E-02
488.0	-.321E+00	.591E+02	.590E+02	-.128E+00	.127E-02
490.0	.234E+01	.591E+02	.597E+02	.642E+00	.127E-02
492.0	.116E+01	.591E+02	.600E+02	.918E+00	.127E-02
494.0	-.636E+00	.591E+02	.597E+02	.585E+00	.127E-02
496.0	.189E+01	.591E+02	.602E+02	.106E+01	.127E-02
498.0	-.139E+01	.591E+02	.596E+02	.446E+00	.127E-02
500.0	.106E+01	.591E+02	.598E+02	.666E+00	.127E-02

X2	X2E	X3	AVXNT	AVX1E	SMOOTH
.606E-01	.593E-01	.220E-02	.239E-01	.538E-02	.677E+02
.124E+00	.122E+00	.486E-02	.371E-01	.112E-01	.678E+02
.665E-01	.653E-01	.183E-02	.320E-01	.146E-01	.678E+02
-.230E-01	-.242E-01	-.239E-02	.191E-01	.135E-01	.681E+02
.528E-01	.515E-01	.126E-02	.322E-01	.166E-01	.684E+02
.697E-01	.684E-01	.191E-02	.382E-01	.209E-01	.683E+02
-.691E-01	-.704E-01	-.455E-02	.178E-01	.182E-01	.690E+02
-.341E-01	-.354E-01	-.255E-02	.221E-01	.171E-01	.690E+02
.168E-01	.155E-01	-.167E-04	.305E-01	.187E-01	.692E+02
.143E-01	.130E-01	-.127E-03	.318E-01	.203E-01	.691E+02
.423E-01	.410E-01	.115E-02	.383E-01	.235E-01	.692E+02
.964E-01	.951E-01	.350E-02	.508E-01	.298E-01	.692E+02
.914E-01	.902E-01	.296E-02	.558E-01	.363E-01	.690E+02
.993E-01	.980E-01	.305E-02	.634E-01	.438E-01	.688E+02
.321E-01	.308E-01	-.276E-03	.591E-01	.483E-01	.687E+02
-.419E-01	-.432E-01	-.360E-02	.514E-01	.493E-01	.688E+02
-.239E-01	-.252E-01	-.246E-02	.561E-01	.511E-01	.688E+02
-.110E-01	-.123E-01	-.165E-02	.607E-01	.536E-01	.688E+02
-.232E-01	-.245E-01	-.206E-02	.615E-01	.554E-01	.688E+02
-.129E+00	-.130E+00	-.667E-02	.460E-01	.519E-01	.691E+02
-.113E+00	-.115E+00	-.535E-02	.461E-01	.485E-01	.691E+02
-.103E+00	-.104E+00	-.438E-02	.453E-01	.452E-01	.690E+02
-.906E-01	-.919E-01	-.344E-02	.444E-01	.420E-01	.690E+02
-.794E-01	-.807E-01	-.262E-02	.433E-01	.388E-01	.689E+02
-.230E-01	-.243E-01	.174E-03	.495E-01	.380E-01	.690E+02
-.105E+00	-.107E+00	-.358E-02	.349E-01	.329E-01	.692E+02
-.765E-01	-.777E-01	-.194E-02	.347E-01	.286E-01	.691E+02
-.143E+00	-.144E+00	-.476E-02	.197E-01	.206E-01	.692E+02
-.582E-01	-.595E-01	-.506E-03	.255E-01	.159E-01	.691E+02
-.325E-01	-.338E-01	.704E-03	.246E-01	.121E-01	.690E+02
.107E-01	.947E-02	.260E-02	.271E-01	.100E-01	.690E+02
.128E-01	.116E-01	.246E-02	.245E-01	.789E-02	.690E+02
.966E-02	.838E-02	.209E-02	.212E-01	.553E-02	.690E+02
.316E-01	.303E-01	.290E-02	.217E-01	.413E-02	.689E+02
.839E-01	.826E-01	.500E-02	.278E-01	.524E-02	.690E+02
.869E-01	.857E-01	.469E-02	.284E-01	.676E-02	.690E+02
.224E-01	.211E-01	.134E-02	.192E-01	.556E-02	.692E+02
.804E-01	.792E-01	.385E-02	.267E-01	.716E-02	.693E+02
-.622E-01	-.635E-01	-.296E-02	.579E-02	.241E-02	.700E+02
.188E-01	.175E-01	.975E-03	.138E-01	.120E-02	.701E+02
.159E-01	.146E-01	.754E-03	.120E-01	-.774E-04	.701E+02
.934E-01	.921E-01	.420E-02	.223E-01	.230E-02	.703E+02
.314E-01	.301E-01	.101E-02	.147E-01	.222E-02	.704E+02
.216E-01	.203E-01	.473E-03	.131E-01	.185E-02	.704E+02
.149E-01	.136E-01	.126E-03	.117E-01	.131E-02	.704E+02
.825E-01	.812E-01	.318E-02	.213E-01	.395E-02	.706E+02
.985E-01	.972E-01	.362E-02	.260E-01	.770E-02	.705E+02
.573E-01	.560E-01	.142E-02	.233E-01	.101E-01	.705E+02
.929E-01	.916E-01	.290E-02	.308E-01	.143E-01	.705E+02
.258E-01	.246E-01	-.395E-03	.251E-01	.161E-01	.705E+02
.406E-01	.393E-01	.310E-03	.292E-01	.187E-01	.705E+02

Case d)

P MATRICES



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 1.00E+01 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

Case d)

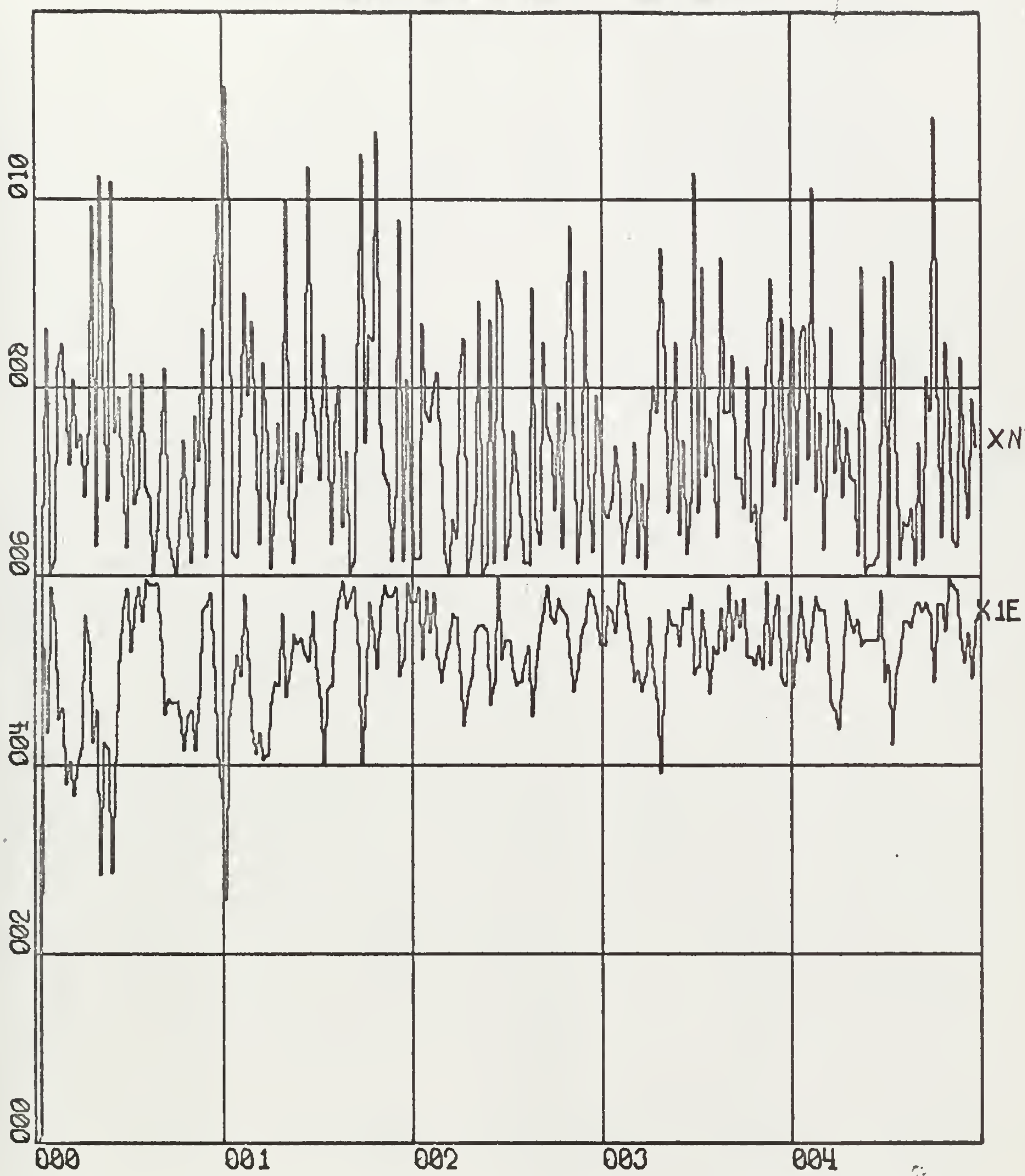
Averages and variances
of
measurement noise and filter error

X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 1.00E+00 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

Case d)

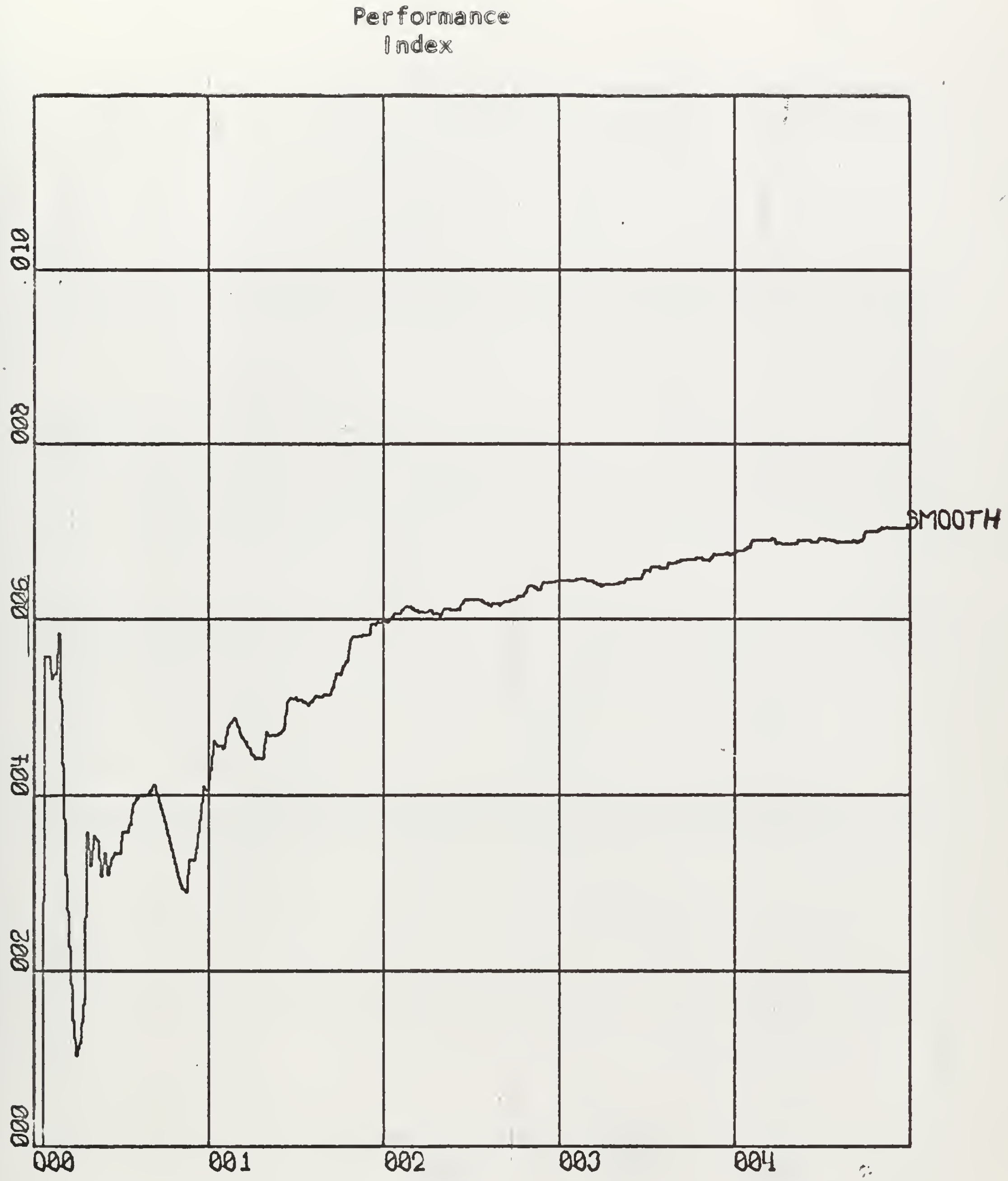
Absolute values
of
noise measurement and filter error



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 2.00E+00 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

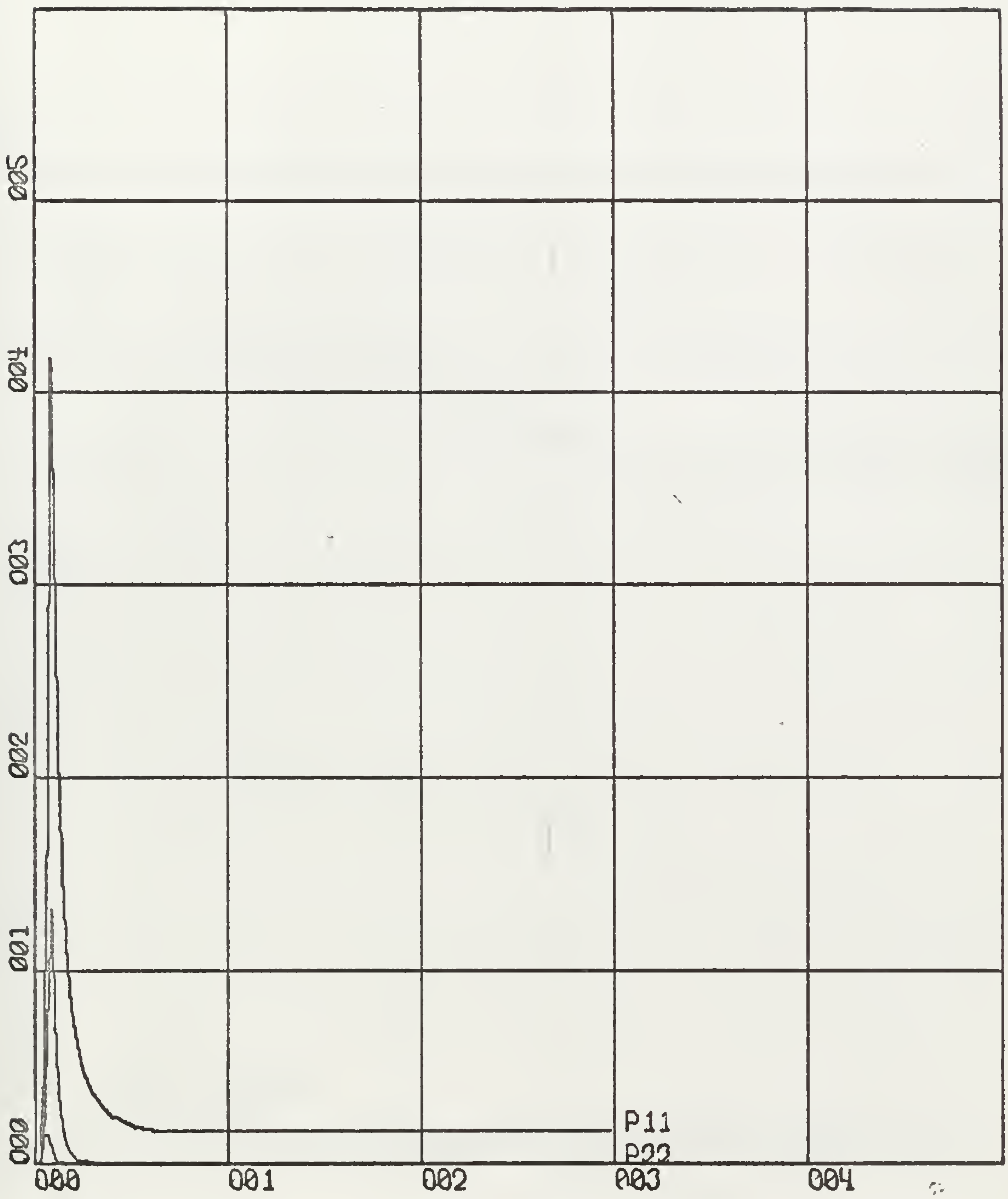
Case d)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 2.00E+01 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

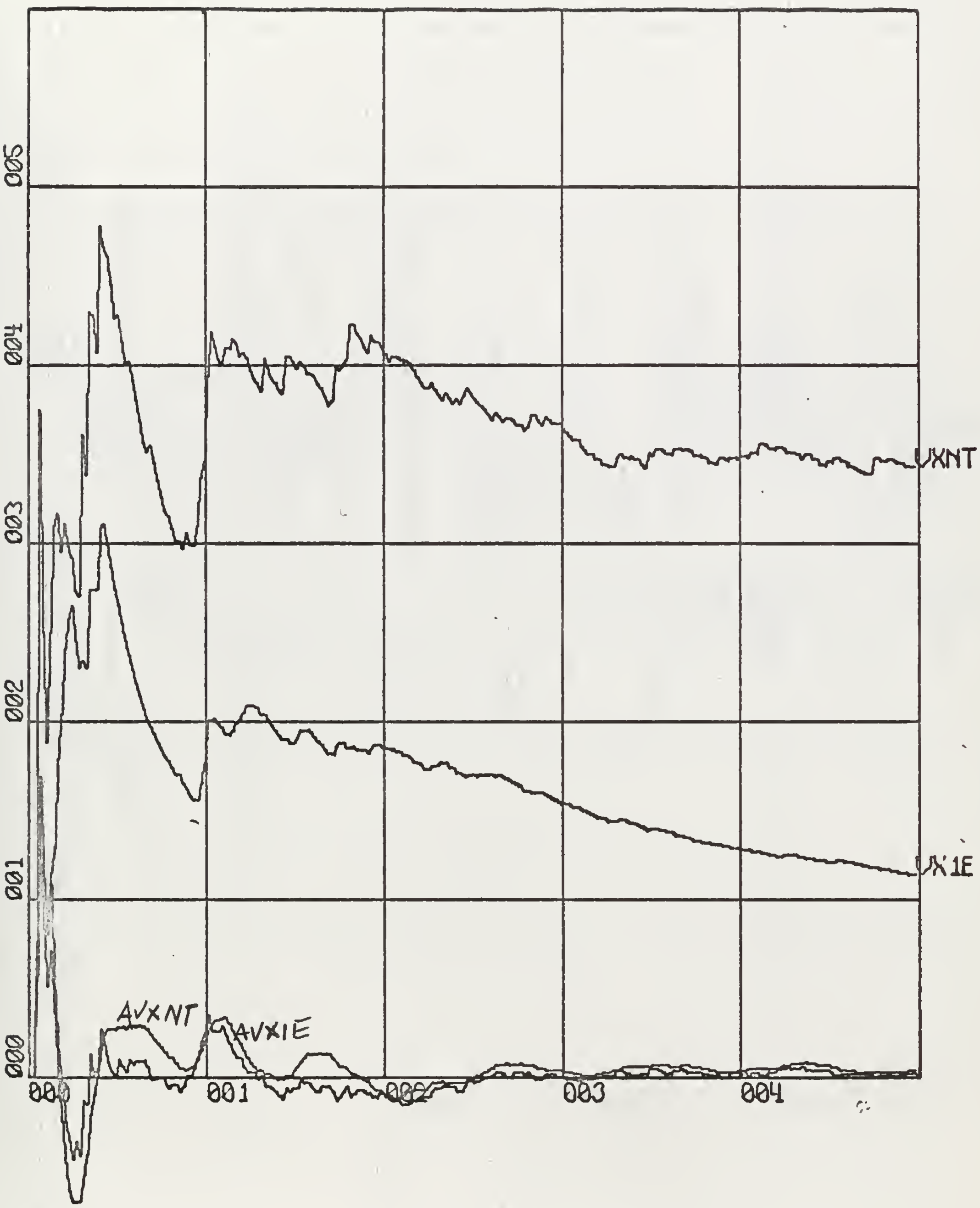
Case b)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 1.00E+01 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

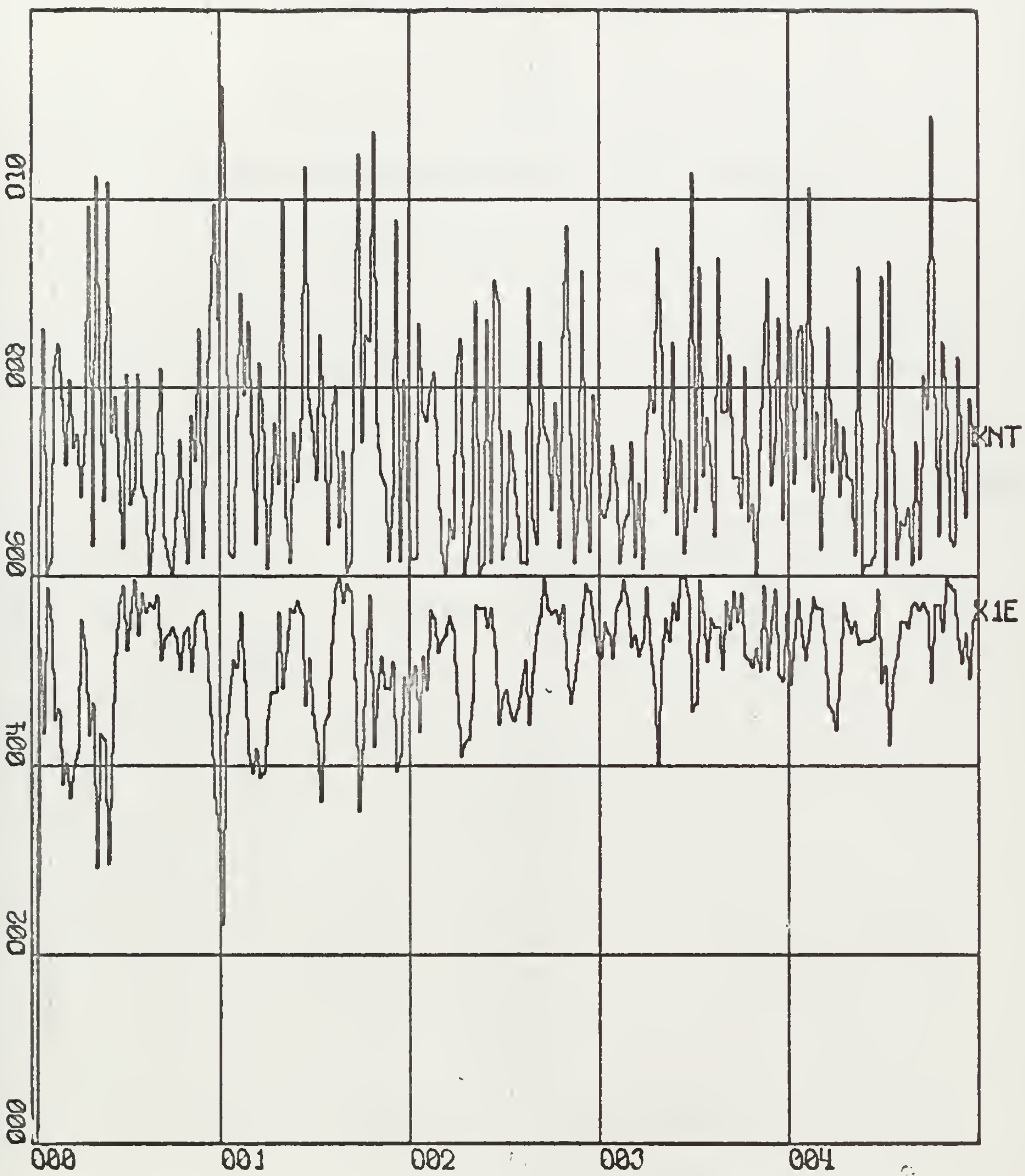
Case b)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 1.00E+00 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

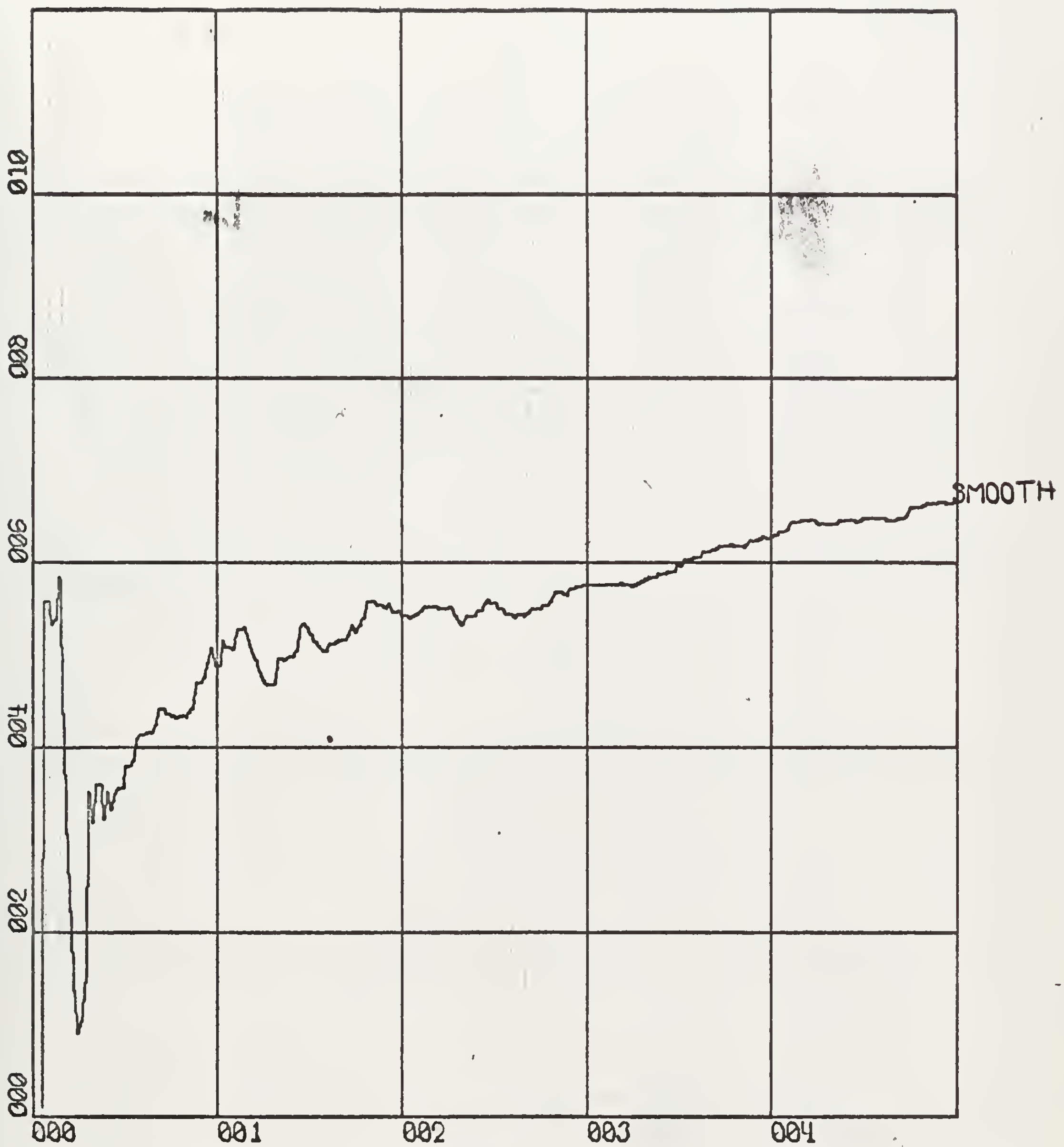
Case b)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 2.00E+00 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

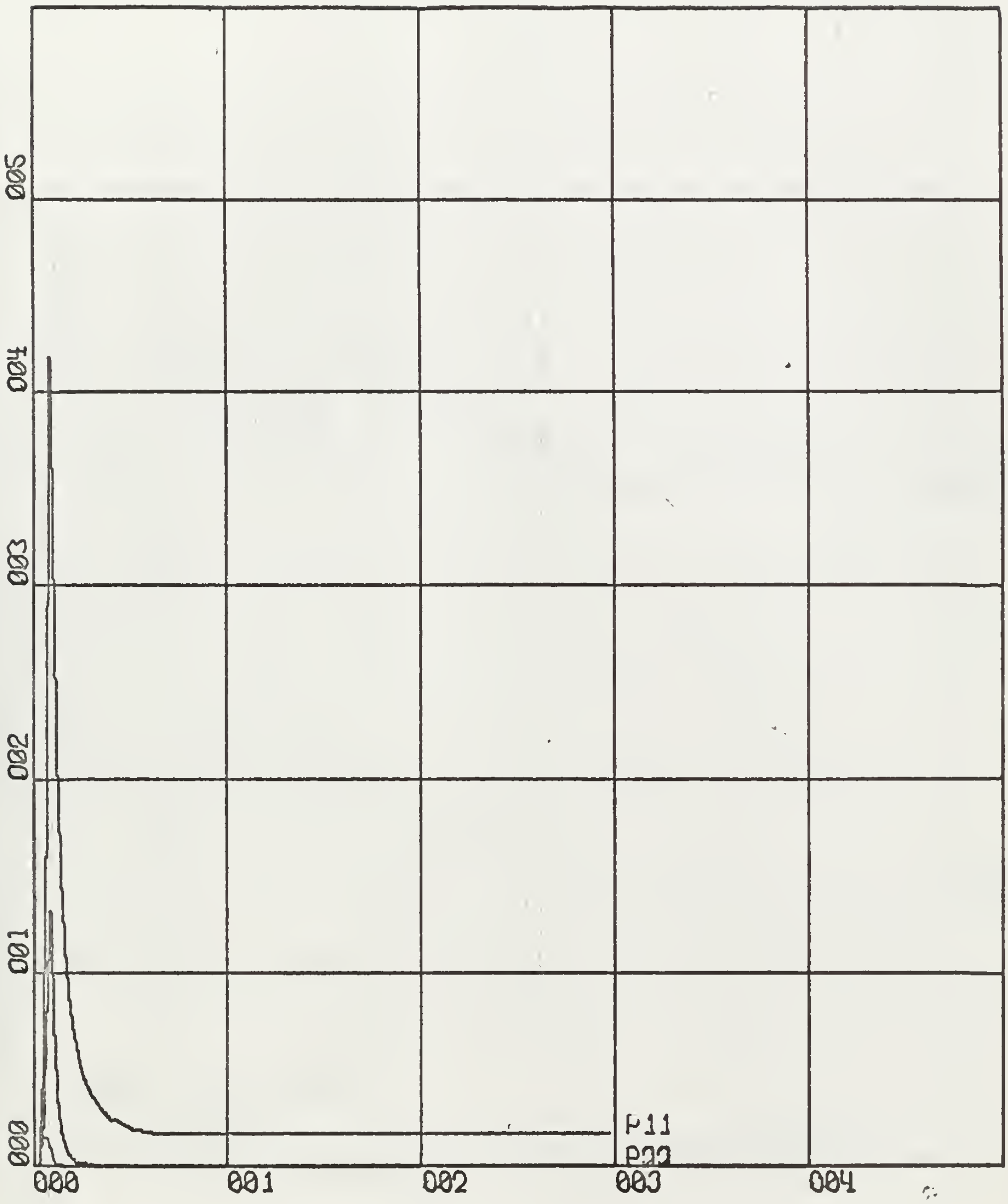
Case b)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 2.00E+01 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

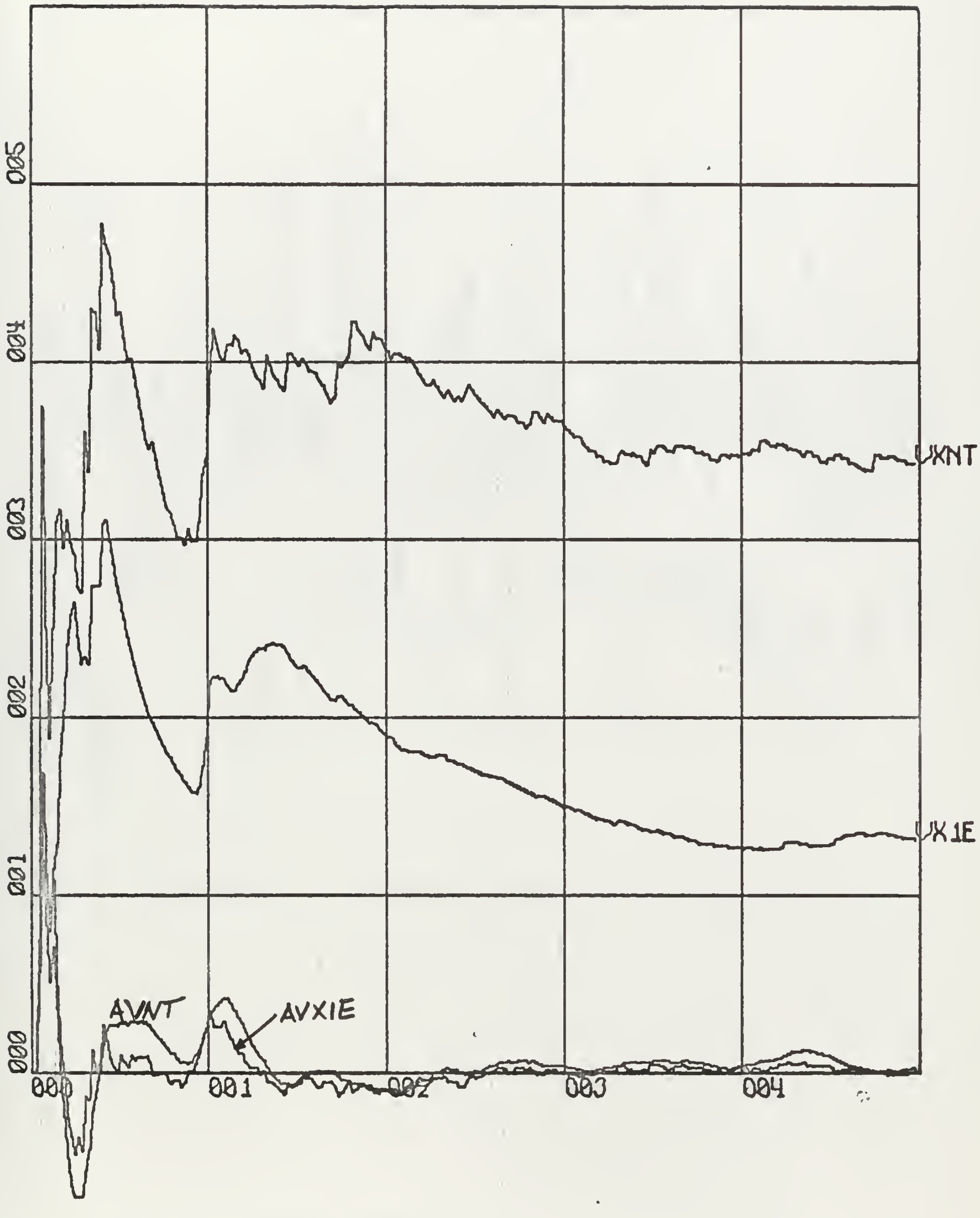
Case a)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 1.00E+01 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

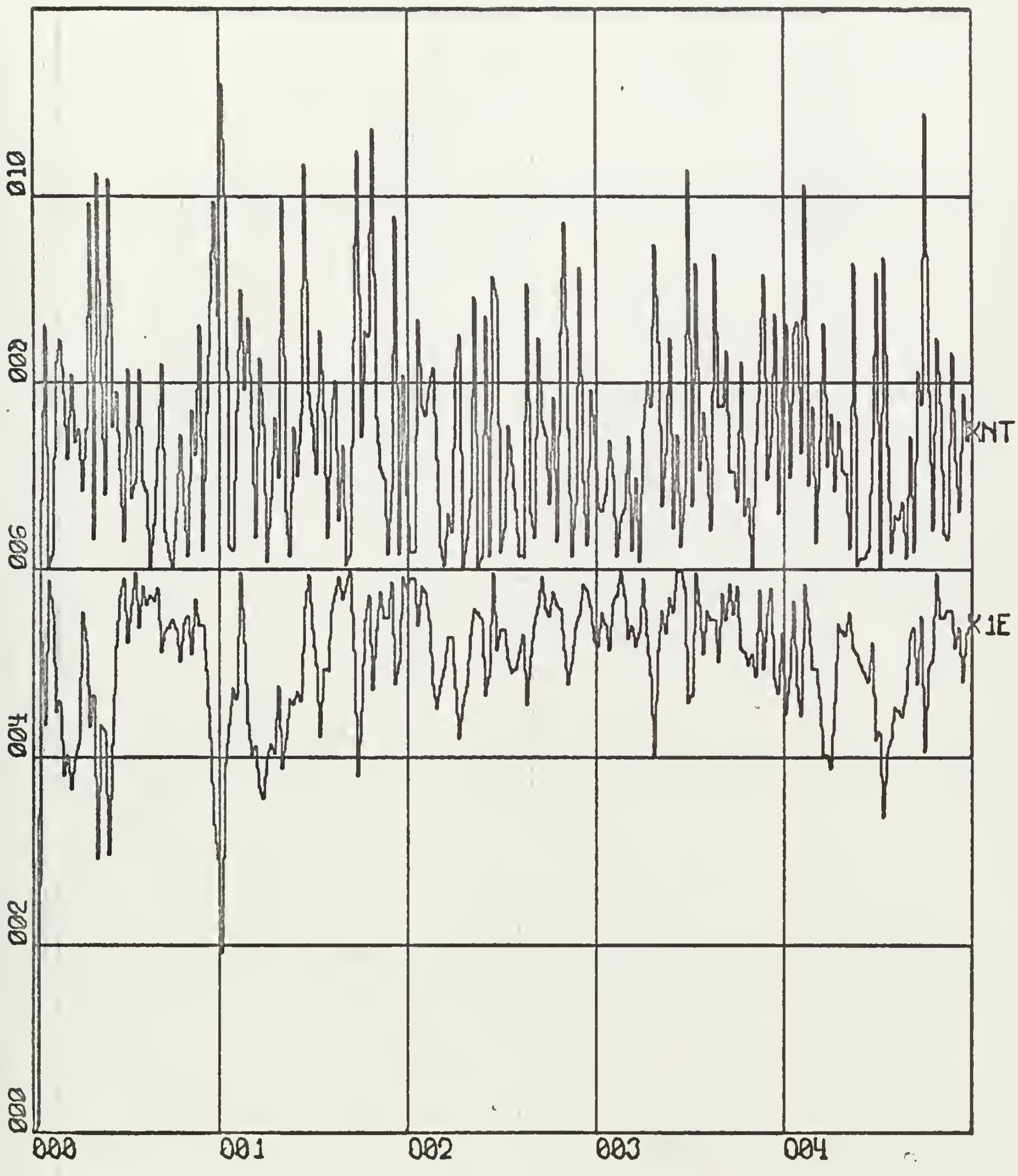
Case a)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 1.00E+00 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

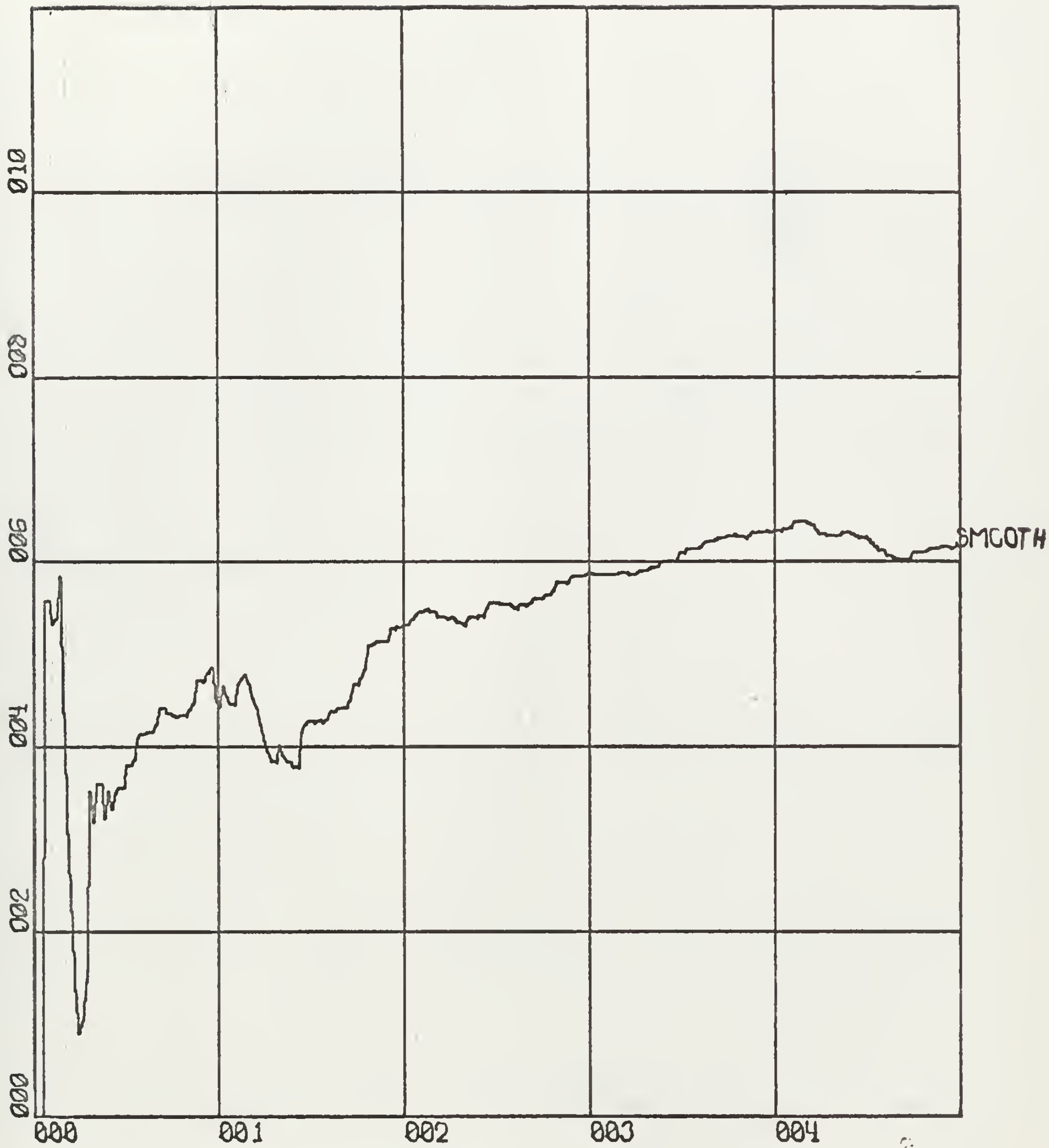
Case a)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 2.00E+00 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

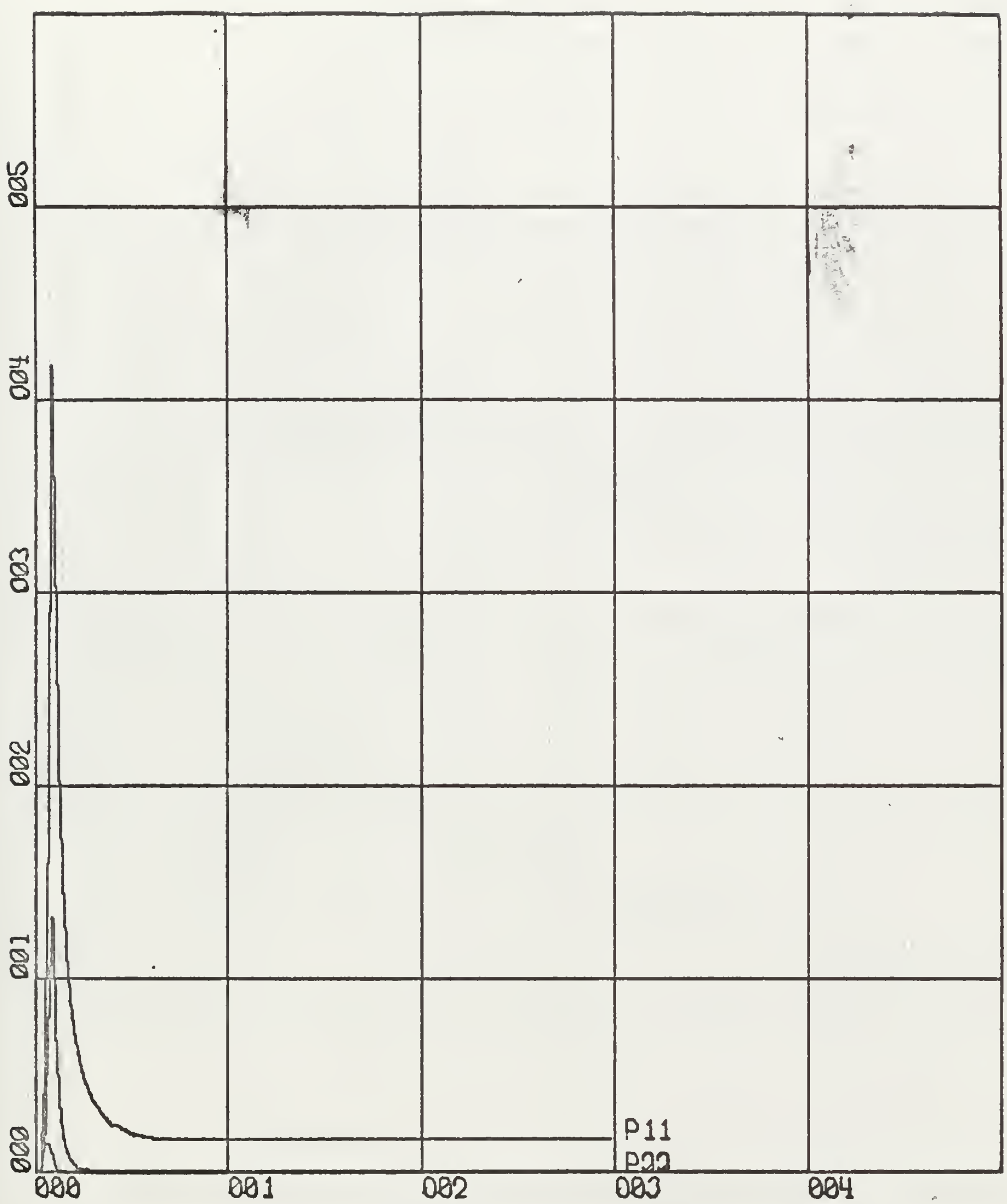
Case a)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 2.00E+01 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

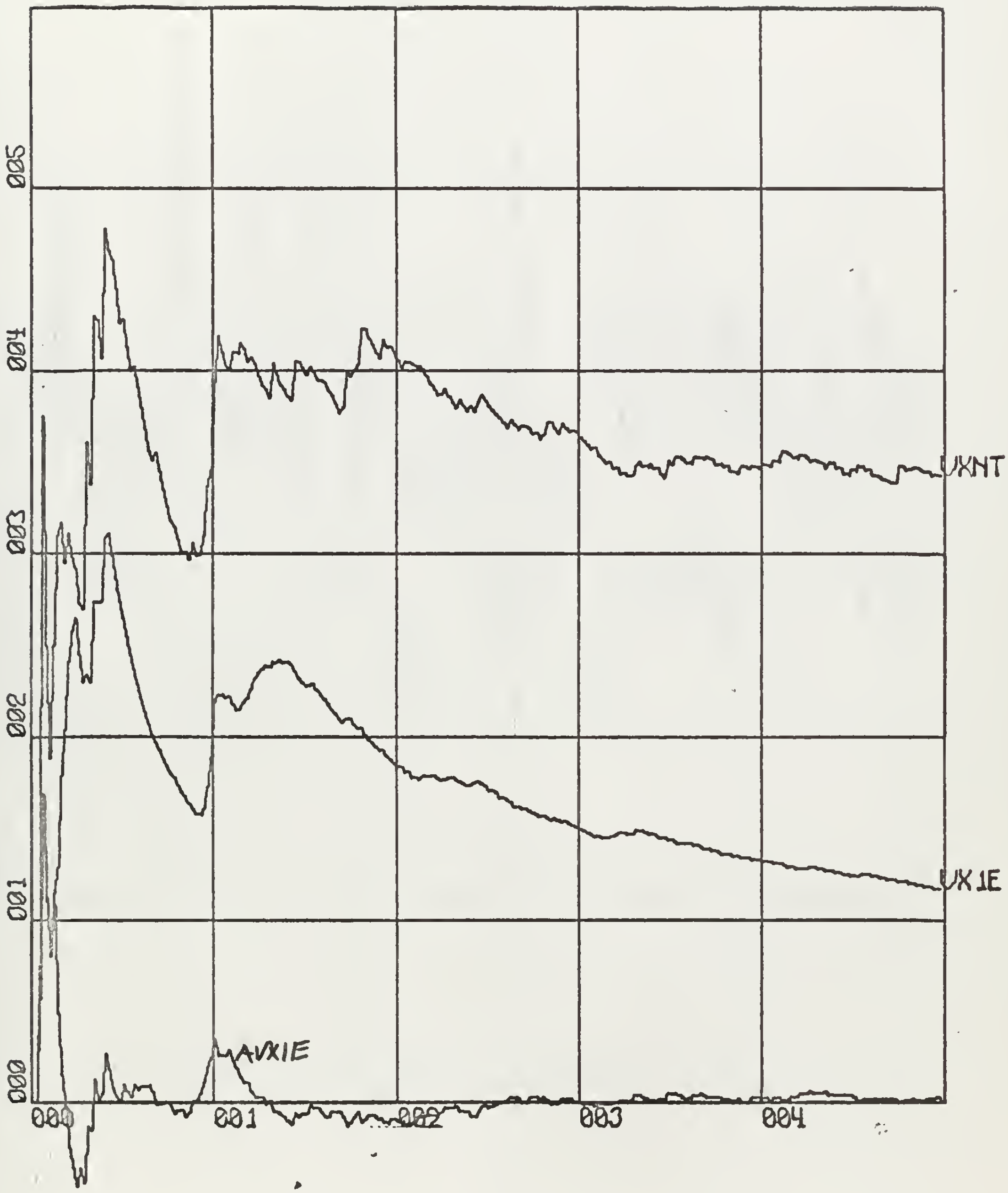
Case c)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 1.00E+01 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

Case c)

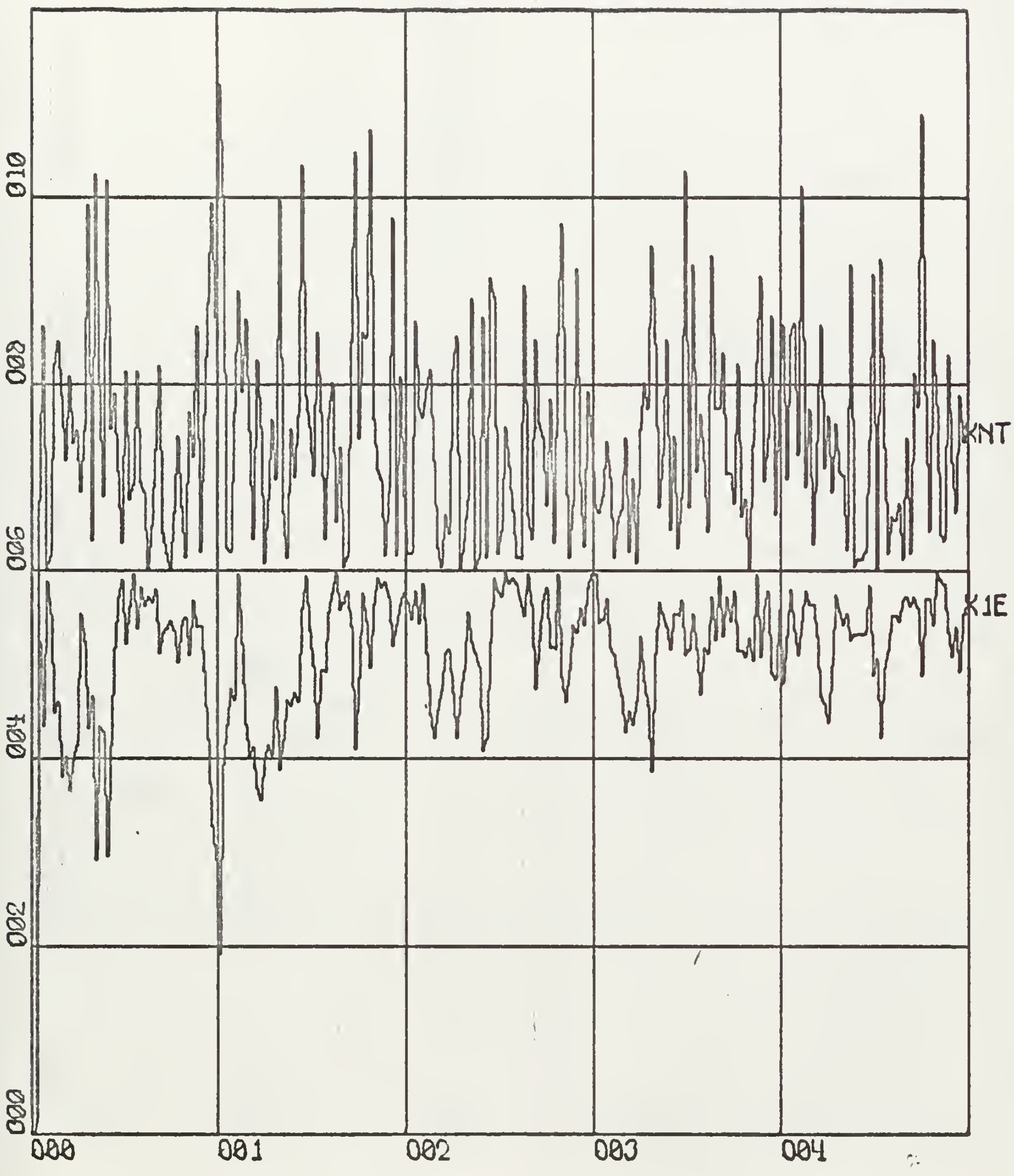


X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 1.00E+00 UNITS/INCH.

CARLOS P. SIMOES MS THESIS

OPTIMAL FILTERING OF RADAR DATA

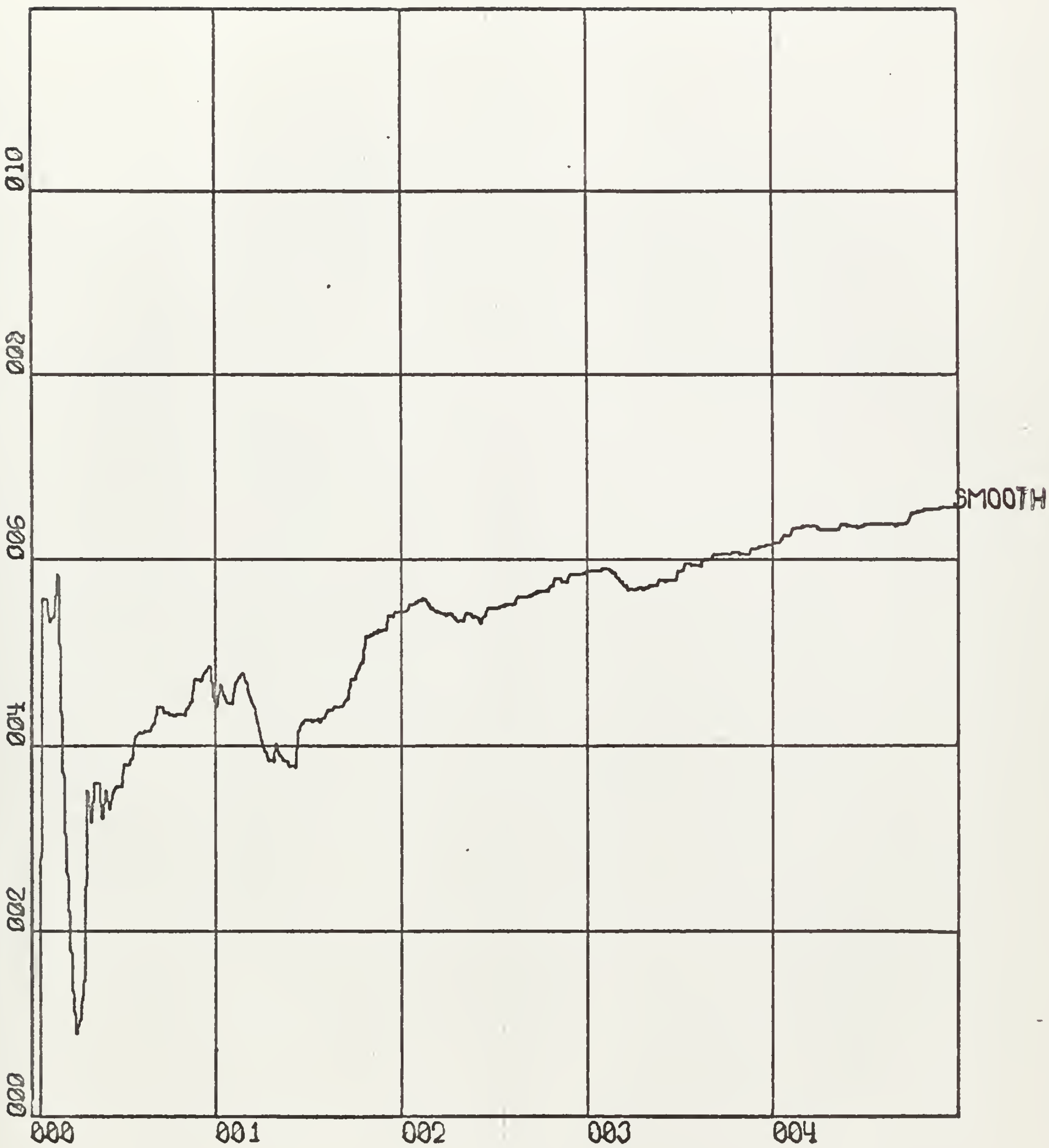
Case c)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 2.00E+00 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA

Case c)



X-SCALE = 1.00E+02 UNITS/INCH.
Y-SCALE = 2.00E+01 UNITS/INCH.

CARLOS P. SIMOES MS THESIS
OPTIMAL FILTERING OF RADAR DATA